ver.4.2, 15 Dec., 2005

On Thurston's inequality for spinnable foliations

Dedicated to Professor Shigeyuki Morita on the occasion of his 60th birthday

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1 Introduction

In this paper, we discuss *Thurston's inequalities for foliations on 3-manifolds*, which are closely related to *Thurston-Bennequin's inequalities for contact structures* (on 3-manifolds). They hold if the structures are convex in some sense.

As the theory of foliations and contact topology have exhibited so many similarities, Eliashberg and Thurston [ETh] developed the theory of *confoliations* to clarify the reason as well as to unify two theories to a certain extent. The relation between the above inequalities is one of its main subjects.

The main result of this paper is to show the violation of Thurston's inequality for spinnable foliations (*i.e.*, a foliation associated with an open book decomposition) under certain conditions on the monodromy (Theorem c, C), as well as the (non-)vanishing of the Euler class of the tangent bundle to those foliations (Theorem a, A, Proposition b, B). These results are stated in §2 and proved in §3. In §4, some application to the mapping class of the monodromy is given, by passing through the relative inequality from the absolute one. One of the key ideas in this paper is to introduce the notion of being (non-)*skinny* for the monodromy. This notion is described along the statements of the main results. In the rest of this section we review Thurston's inequality and some relevant results concerning monodromy of spinnable foliations.

1.1 Thurston's Inequality

Let $\mathcal{F}$ be a transversely (and therefore also tangentially) oriented codimension one foliation on a closed oriented 3-manifold M. Assume that $\mathcal{F}$ has no Reeb components. Then

*Supported by IHÉS(EPDI)-JSPS fellowship.
†Supported by Grant-in-Aid for Scientific Research 16540080
2000 *Mathematics Subject Classification.* Primary 57R20; Secondary 57R30;
Keywords and phrases. foliation, Euler class, contact structure, Thurston's inequality, open book decomposition

for any embedded closed oriented surface Σ of genus $g > 0$, Thurston showed that the following inequality holds:

Absolute Thurston's Inequality (cf. [Th])

$$|\langle e(T\mathcal{F}), [\Sigma]\rangle| \leq |\chi(\Sigma)| = 2g - 2,$$

where $e(T\mathcal{F})$ denotes the Euler class of the 2-plane field $T\mathcal{F}(\subset TM)$ tangent to the foliation $\mathcal{F}$, $[\Sigma]$ is the homology class represented by the closed surface Σ and $\chi(\Sigma)$ denotes the Euler characteristic of Σ.

This inequality can be expressed in terms of the (dual) Thurston norm $\|\cdot\|_{\mathrm{Th}}$ for homology or $\|\cdot\|_{\mathrm{Th}^*}$ for cohomology as follows (for these norms see [Th]) :

$$|\langle e(T\mathcal{F}), z\rangle| \leq \|z\|_{\mathrm{Th}} \ \ (\forall z \in H_2(M;\mathbb{Z}))$$

or equivalently

$$\|e(T\mathcal{F})\|_{\mathrm{Th}^*} \leq 1.$$

We have a much more refined version of this inequality, the relative version. Let Σ be any Seifert surface such that the oriented boundary $L = \partial\Sigma$ is a positive transverse link to $\mathcal{F}$ *w.r.t.* its transverse orientation. Take any non-zero section X of the restriction $T\mathcal{F}|\Sigma$ of $T\mathcal{F}$ to Σ and let L^X denote the shift of L along $X|L$. Consider the linking number $\mathrm{lk}(L, L^X)$ between L and L^X, which is also regarded as the relative Euler number $-\langle e(T\mathcal{F}), [\Sigma, L]\rangle$ under a suitable boundary condition. Now we again assume that $\mathcal{F}$ has no Reeb components. Then the relative version holds as follows.

Relative Thurston's Inequality (cf. [Th]) $\quad \mathrm{lk}(L, L^X) \leq -\chi(\Sigma)$.

For the most of foliations (very likely but quite few exceptions), eventually the approval of the relative inequality is stronger than that of the absolute one (see §4).

As is mentioned above, these inequalities have their complete analogues in contact topology. Simply replacing $T\mathcal{F}$ with an oriented contact plane field ξ in Thurston's inequalities, we obtain so called *Thurston-Bennequin's inequalities* for oriented contact structures. In the contact case, because the relative inequality is definitely stronger than the absolute one, we refer to the relative one simply as Thurston-Bennequin's inequality. The first contact structure for which the inequality was proven is the standard contact structure on S^3, which is due to Bennequin [B]. (In this case usually we call the inequality and its variants *Bennequin's inequalities.*) In general, the inequality holds if and only if the structure is tight ([B], [E]).

Bennequin first proved relative Thurston's inequality for the standard Reeb foliation, in order to deduce his inequality for the standard contact structure on S^3. Moreover, as is mentioned in the next subsection, now we know a lot of other foliations which satisfy relative Thurston's inequality, even though they have Reeb components.

We would like to begin the study of Thurston's inequalities for foliations with Reeb components in this paper. The main result is *the violation of absolute Thurston's inequality for a certain class of foliations.* As a consequence, we see that a certain class of diffeomorphism on a surface is isotopic neither to a product of right-handed Dehn twists nor to that of left-handed Dehn twists.

1.2 Convergence of Contact Structures to Foliations

Now let us explain our motivation from the contact topological view point.

The phenomena of the convergence of contact structures to foliations as plane fields were recognized and well studied in the work of confoliations [ETh] (see also [Mi1]). Especially, if a family of tight contact structures converges to a foliation, (relative as well as absolute) Thurston's inequality holds. Here we should remark that the converse does not hold, *i.e.*, there exists a family of overtwisted contact structures which converges to the standard Reeb foliation [Mi2].

Thurston and Winkelnkemper [ThW] found a canonical way to construct a contact structure from a spinnable structure (=an open book decomposition). It is also known that an oriented closed 3-manifold is the boundary of a compact Stein surface if and only if it admits a spinnable structure whose monodromy is a product of right-handed Dehn twists ([LoP]). Combined with this, Mori's result on Thurston-Winkelnkemper's construction implies the following theorem.

Theorem ([LoP], [Mo]) *Suppose ξ is obtained from a spinnable structure $\mathcal{S}$ on M with Thurston-Winkelnkemper's construction. Assume that the monodromy diffeomorphism of $\mathcal{S}$ can be written as a product of only right-handed Dehn twists. Then Thurston-Bennequin's inequality holds for ξ.*

Also a foliation, called a *spinnable foliation*, is naturally associated with a spinnable structure. See the following sections for detailed construction. Thurston-Winkelnkemper's contact structure has an isotopic family which converges to this spinnable foliation. As he remarked in [Mo], Mori's result implies that Thurston's inequalities hold for such a spinnable foliation even though it admits Reeb components (see §4 below). Therefore, the following problems naturally arise.

Problems 1) *Determine the class of foliations with Reeb components for which absolute Thurston's inequality holds.*
2) *Determine the subclass of* 1) *for which relative Thurston's inequality holds.*
3) *Prove the inequality for the class of* 2) *directly in the framework of* 3*-dimensional topology, like Bennequin's work* [B]*, without passing through global analytic methods on* 4*-manifolds* (*e.g. moduli of pseudo-holomorphic curves, Seiberg-Witten theory etc.*)*.*

Our aim in this paper is to study a part of Problem 1) for spinnable foliations.

2 Statement of Results

Let M be a closed oriented 3-manifold. A *spinnable structure* (or an *open book decomposition*) $\mathcal{S} = (L, F, \pi)$ on M is a fibred link L in M with a specified fibration. In other words, the *axis* of a spinnable structure is an oriented link $L = \cup_i L_i$ in M and the spinnable structure is nothing but a fibration $\pi : M - L \to S^1$ which behaves nicely near the axis. Precisely, with respect to a framing $S^1 \times D^2(\ni (\theta, x)) \to N(L_i)$ of a tubular neighbourhood of L_i, the projection $\pi|N(L_i) - L_i$ is of the form $\pi(\theta, x) = x/|x| = \omega \in \mathbb{R}/\mathbb{Z}$. Then a spinnable structure can be expressed by a *monodromy diffeomorphism* $\varphi : F \to F$ as $M - \mathrm{int}N(L) = F \times [0,1]/\varphi$ where F is the Seifert surface $F = \pi^{-1}(0) \cap (M - \mathrm{int}N(L))$ $(0 \in \mathbb{R}/\mathbb{Z})$, the front surface $F \times \{1\}$ is attached to the back surface $F \times \{0\}$ by $\varphi : F \times \{1\} \to F \times \{0\}$, and φ is assumed to be supported in $\mathrm{int}F$. Let $\mathcal{S}_\varphi$ denote the spinnable structure equipped with a fixed monodromy φ.

Here we have to remark that the orientation of the link L determines the orientations of the normal disk D^2, the base space $S^1 (\approx \partial D^2)$ and the fibre F, respectively. Then the oriented boundary ∂F is parallel to L in the same direction.

2.1 Spinnable Foliations

Given a spinnable structure $\mathcal{S}_\varphi$, we construct a depth one foliation $\mathcal{F}_\varphi$ with Reeb components on a tubular neighbourhood R of the axis L and non-compact leaves obtained from the fibres turbulized along the border leaves ∂R (see Figure 1).

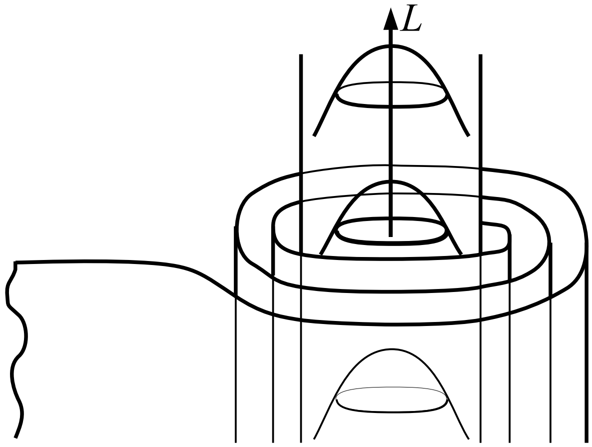

Figure 1: Spinnable Foliation

Without taking care of orientations, we might have essentially $2^{2\sharp\pi_0(L)}$ possibilities for $\mathcal{F}_\varphi$. Namely, for each component of L, we might have two choices to place the Reeb component (in which direction the interior leaves are convex) and two choices for the orientation of its meridian of the component of L along which the fibres spiral into the border leaf. We fix a unique choice as follows. Suppose that the interior leaves of $\mathcal{F}_\varphi|R$ are convex above *w.r.t.* the orientation of L. Then the outward normal of the border leaves ∂R is positive *w.r.t.* the L-induced orientation of $\mathcal{F}_\varphi|R$. Suppose also that this outward normal agrees with the transverse orientation of each exterior leaf which is originally a fibre of the spinnable structure. That is, when the outside leaves spiral around and come closer to the border leaf, each leaf is supposed to come back slightly below the original position *w.r.t.* its transverse orientation. See Figure 1 and the more precise construction of $\mathcal{F}_\varphi$ in the next section. We call $\mathcal{F}_\varphi$ a *spinnable foliation* associated with $\mathcal{S}_\varphi$.

Remark. If we place the Reeb components upside down on R, the *inward* normal of ∂R will be positive *w.r.t.* the L-induced orientation. Then, assuming this inward normal to be positive *w.r.t.* the exterior fibration, we have to pick up the opposite direction for the fibres to spiral into ∂R. Let $\mathcal{G}_\varphi$ denote the resultant foliation. This second construction is not different from the first one in the following sense. Keeping the orientation of M fixed, we reverse the orientation of the link L. Topologically, the fibration π does not change at all, however, the orientation and the normal orientation of the fibres are reversed. Then, we get a new spinnable structure with the monodromy $-\varphi^{-1} : -F \to -F$. If we apply the first construction to this new spinnable structure $\mathcal{S}_{-\varphi^{-1}}$, the resultant foliation $\mathcal{F}_{-\varphi^{-1}}$ coincides with the second foliation $-\mathcal{G}_\varphi$ with reversed orientation. If we were to confuse

$\mathcal{F}_\varphi$ with $\mathcal{G}_\varphi = -\mathcal{F}_{-\varphi^{-1}}$, the important notions in the present article would not change at all, *e.g.*, the oriented 3-manifold M, the monodromy φ being written as a product of right-handed Dehn twists, the vanishing of $e(T\mathcal{F}_\varphi)(= e(T\mathcal{G}_\varphi))$, the (dis-)approval of absolute Thurston's inequality for $\mathcal{F}_\varphi$ *etc.*.

2.2 Results

To state the first result, we assume that the axis of a spinnable structure is connected for a while. Therefore, as a monodromy diffeomorphism φ, we only consider a diffeomorphism of a once punctured compact oriented surface which fixes the boundary. It is well known that such a diffeomorphism can be written as a product of Dehn twists up to isotopy. Figure 2 shows a system of loops $C_0, C_1, \dots, C_{2g}$ along which Dehn twists τ_{C_i}'s generate the mapping class group of the surface (see [Li] and [H]). This set of generators is called the *Dehn-Lickorish-Humphries generators* (*D-L-H generators*, for short).

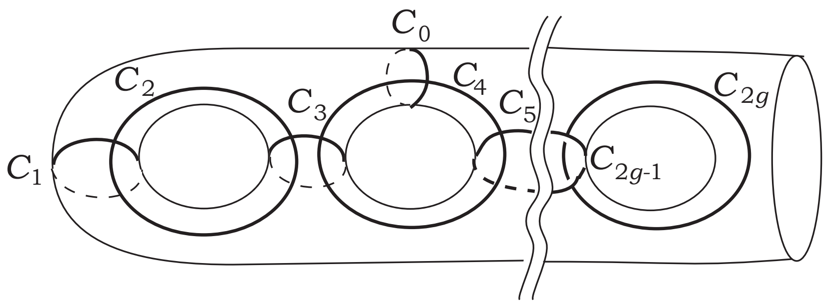

Figure 2: Dehn-Lickorish-Humphries generators

We call a diffeomorphism $\psi : F \to F$ *DLH-skinny w.r.t.* a fixed D-L-H presentation if it is isotopic to a product of Dehn twists along C_i's except one curve which is C_0, C_1 or C_3.

Theorem a *If the monodromy diffeomorphism φ of a spinnable foliation $\mathcal{F}_\varphi$ admits a D-L-H presentation w.r.t. which φ is DLH-skinny, then the Euler class $e(T\mathcal{F}_\varphi)$ vanishes.*

This is a special case of the following generalization where ∂F is not necessarily connected. The proof is given for Theorem a in the spirit of proving Theorem A.

Theorem A *Suppose that the monodromy diffeomorphism φ is given as a product*

$\varphi = \prod_{k=1}^{l} \tau_{\gamma_k}^{j_k}$ *of Dehn twists along simple closed curves* γ_k*'s. If* γ_k*'s satisfy the following condition* (S)*, the Euler class* $e(T\mathcal{F}_\varphi)$ *of the spinnable foliation* $\mathcal{F}_\varphi$ *vanishes.*

Condition (S) *The curves* $\cup_{k=1}^{l}\gamma_k$ *may have transverse intersections but has no triple points, and each connected component of* $F - \cup_{k=1}^{l}\gamma_k$ *contains at least one component of* ∂F*.*

We call a diffeomorphism φ simply *skinny* if φ satisfies the condition in Theorem A.

The condition (S) implies that one may assume the curves $\bigcup_{k=1}^{l}\gamma_k$ on a fibre $F \times \{0\}$ to be Legendrian *w.r.t.* Thurston-Winkelnkemper's contact structure associated with $\mathcal{S}_\varphi$.

Now, we return to the original setting, namely, F has only one boundary component and the mapping classes are generated by the D-L-H generators. Then it is clear that if the Euler class is trivial or of finite order, absolute Thurston's inequality trivially holds. Therefore, it is sufficient for our purpose to consider the case where the Euler class is of infinite order. Further, we restrict ourselves to the following situation. First let us fix a D-L-H presentation of the mapping class of the monodromy φ.

Condition (i) In the fixed D-L-H presentation, the monodromy φ is presented as

$$\varphi = \tau_{C_0}^{j_0}\tau_{C_1}^{j_1}\tau_{C_3}^{j_3} \cdot \prod_{k=4}^{l} \tau_k^{j_k}, \quad j_0 j_1 j_3 \neq 0 \quad (\tau_k = \tau_{C_{i_k}},\, i_k \in \{5, 6, \dots, 2g\},\, k = 1, \dots, l).$$

Namely τ_{C_2} and τ_{C_4} do not appear and τ_{C_0}, τ_{C_1}, and τ_{C_3} do appear.

Under this condition, the generators τ_{C_0}, τ_{C_1} and τ_{C_3} commute with any other generators. Then we have the following criterion for the Euler class being of infinite order.

Proposition b *Assume that the monodromy* φ *of a spinnable foliation* $\mathcal{F}_\varphi$ *satisfies the conditions* (i)*. Then, the Euler class* $e(T\mathcal{F}_\varphi)$ *is of infinite order if and only if the following condition* (ii) *is satisfied.*

Condition (ii) $\quad \dfrac{1}{j_0} + \dfrac{1}{j_1} + \dfrac{1}{j_3} = 0.$

Theorem c *Assume the conditions* (i) *and* (ii) *for the monodromy* φ *of a spinnable foliation* $\mathcal{F}_\varphi$*. Then absolute Thurston's inequality does not hold, i.e., there exists an embedded closed oriented surface* Σ *with* $|\langle e(T\mathcal{F}_\varphi), [\Sigma]\rangle| > |\chi(\Sigma)|$*.*

In order to prove Theorem c, we will find a closed oriented surface in the exterior of the axis on which the evaluation of the Euler class is not zero (Proposition b). Then one can perform surgery on this surface to get another closed oriented surface with far less genus, which shows the violation of Thurston's inequality.

We can again generalize (the "*if*" part of) Proposition b and Theorem c as follows. Here the surface F is allowed to have more than one boundary components.

Condition (I) (1) *There exist a compact connected subsurface* P *of genus* 0 *in* $\mathrm{int}F$ *with boundary* $\partial P = \bigcup_{i=1}^{m} \gamma_i$ $(m \geq 3)$ *and a disjoint family of simple closed curves* $t_{i,i+1}$ $(i = 1, 2, 3, ..., m,$ *where* $m+1$ *is taken as* $1)$ *such that the intersection of* P *and each* $t_{i,i+1}$ *is an arc joining* γ_i *and* γ_{i+1}.
(2) *The monodromy* φ *is given as the product* $\varphi = \prod_{i=1}^{m} \tau_{\gamma_i}^{j_i} \cdot \prod_{k=m+1}^{l} \tau_{\delta_k}^{j_k}$, *where each* δ_k *denotes a simple closed curve disjoint from* $P \cup t_{1,2} \cup \cdots \cup t_{m,1}$ $(k = m+1, \cdots, l)$.
(3) $j_1 \cdot \cdots \cdot j_m \neq 0$.
(4) *The connected component of* $F - \bigcup_{k=m+1}^{l} \delta_k$ *containing* P *meets the boundary* ∂F, *i.e., there exists an arc which joins* ∂F *and* $P \cup t_{1,2} \cup \cdots \cup t_{m,1}$ *without meeting* $\bigcup_{k=m+1}^{l} \delta_k$.

Condition (II) $\dfrac{1}{j_1} + \cdots + \dfrac{1}{j_m} = 0.$

Proposition B *Under the conditions* (1), (2), *and* (3) *of* (I), $e(T\mathcal{F}_\varphi)$ *is of infinite order if the condition* (II) *is satisfied.*

Theorem C *Under the conditions* (I) *and* (II), *absolute Thurston's inequality does not hold for* $\mathcal{F}_\varphi$.

Conditions (i) and (I) are geometric condition to assure that the monodromy is non-skinny (for a fixed D-L-H presentation). Conditions (ii) and (II) further assures algebraically that the Euler class is in fact alive. Then they all together implies the followings.

Corollary d *Conditions* (i) *and* (ii) *imply that the monodromy is not DLH-skinny for any D-L-H presentation.*

Corollary D *The conditions* (I) *and* (II) *for the monodromy* φ *implies that* φ *can not be presented as in Theorem A.*

3 Proofs

Let M be a closed oriented 3-manifold which has a spinnable structure $\mathcal{S} = (L, F, \pi)$ with monodromy $\varphi : F \to F$ and $\mathcal{F}_\varphi$ its spinnable foliation on M. Basically we give proofs only for Theorem a, Proposition b, and for Theorem c. Therefore, we assume that L and ∂F are knots in M. (Once they are understood, we believe it fairly straightforward to generalize them to those of Theorem A, Proposition B, and of Theorem C.) We fix a diffeomorphism $M - \mathrm{int}N(L) \approx M_\varphi$, where $N(L)$ denotes a closed tubular neighbourhood of the axis L and $M_\varphi = F \times [0,1]/\varphi$ denotes the mapping torus of φ , so that the restricted foliation $\mathcal{F}_\varphi | M_\varphi$ is the (twisted) product foliation $\{F \times \{\omega\}; \omega \in \mathbb{R}/\mathbb{Z}\}$.

3.1 Spinnable Foliations around L

We will describe the structure of $\mathcal{F}_\varphi | N(L)$, the foliation restricted to $N(L)$. Fix a cylindrical coordinate $(\theta, r, \omega) \in S^1_\theta \times D^2(4)$ for $N(L) \approx S^1_\theta \times D^2(4)$. Here L corresponds to $S^1_\theta \times \{(0, *)\}$ (*i.e.*, the θ-axis), the pair $(r, \omega) \in [0, 4] \times S^1_\omega$ is the polar coordinate for the normal disk of radius 4, both of S^1_θ and S^1_ω are defined as $\mathbb{R}/\mathbb{Z}$, the projection $\pi | (N(L) - L)$ corresponds to respecting ω and ignoring (θ, r), and the Reeb component of $\mathcal{F}_\varphi$ occupies the closed tubular neighbourhood $R = S^1_\theta \times D^2(2)$ of radius 2. Then take smooth decreasing functions $f_i(r)$ on $[0, 5)$ such that

$$\begin{cases} f_i(r) \equiv ((-1)^i + 1)/2 & \text{on} \quad [0, i-1] \\ f_i(r) \equiv ((-1)^i - 1)/2 & \text{on} \quad [i, 5) \end{cases} \qquad (i = 1, 2, 3, 4)$$

and define vector fields

$$\Theta = f_1(r)\frac{\partial}{\partial\theta} + f_2(r)\frac{\partial}{\partial r} \quad \text{and} \quad \Omega = f_3(r)\frac{\partial}{\partial r} + f_4(r)\frac{\partial}{\partial\omega}$$

on $S^1_\theta \times (0, 5) \times S^1_\omega \supset N(L) - L$.

Then, since the bracket $[\Theta, \Omega]$ vanishes on $\mathrm{int}N(L) - L$, we obtain the foliation $\mathcal{F}_\varphi | N(L)$ by integrating $T\mathcal{F}_\varphi | (\mathrm{int}N(L) - L) = \langle \Theta, \Omega \rangle$, the oriented span of Θ and Ω. Here the boundary $\partial N(L)$ and the axis L are perpendicular to $\mathcal{F}_\varphi$. Note that we can also define the foliation $\mathcal{F}_\varphi$ by using a Pfaff form

$$\alpha_0 = \begin{cases} f_2(r)d\theta - f_1(r)f_4(r)dr - f_3(r)d\omega & \text{on} \quad N(L) \\ d\omega & \text{on} \quad M_\varphi \end{cases} ,$$

which satisfies $\alpha_0 \neq 0$, $\alpha_0 \wedge d\alpha_0 \equiv 0$ and $\ker \alpha_0 | (\mathrm{int}N(L) - L) = \langle \Theta, \Omega \rangle$.

3.2 Proof of Theorem a

Even though, as we will see in the next subsection, Theorem a can be proved in a much simpler way, a geometric proof which we need to prove Theorem A is given here.

As above, suppose that $\mathcal{F}_\varphi$ is obtained from a spinnable structure $\mathcal{S} = (L, F, \pi)$ with monodromy $\varphi : F \to F$. By the assumption, φ can be written as the product of Dehn twists where at least one of τ_{C_0}, τ_{C_1} and τ_{C_3} does not appear at all. Thus we can write $\varphi = \prod_{k=1}^{l} \tau_k$ where τ_k denotes the Dehn twist along C_{i_k} and $\{i_1, \ldots, i_l\}$ does not contain at least one of 0, 1 and 3. According to this expression, we divide the mapping torus M_φ into mapping cylinders of τ_k's and consider the "telescope" $(F \times I_1) \cup_{\tau_1} \cdots \cup_{\tau_{l-1}} (F \times I_l)$ where $I_k = \left[\dfrac{k-1}{l}, \dfrac{k}{l}\right]$ and $\cup_{\tau_k}$ denotes the operation of attaching $F \times I_k$ to $F \times I_{k+1}$ by the diffeomorphism $\tau_k : F \times \left\{\dfrac{k}{l}\right\} \to F \times \left\{\dfrac{k}{l}\right\}$. Then we have a natural diffeomorphism

$$M_\varphi \approx (F \times I_1) \cup_{\tau_1} (F \times I_2) \cup_{\tau_2} \cdots \cup_{\tau_{l-1}} (F \times I_l)/\tau_l : F \times \{1\} \to F \times \{0\}$$

where τ_l identifies the two ends $F \times \{1\}$ and $F \times \{0\}$ of the "telescope".

Take a non-vanishing vector field X_j on the annular support of each Dehn twist τ_{C_j} on F which is parallel to the core curve C_j $(j = 0, \ldots, 2g)$. Then, on the union $U_i = \bigcup_{j \neq i} \operatorname{supp} \tau_{C_j} \subset F$ $(i = 0, 1 \,\text{or}\, 3)$, we define a non-vanishing vector field X on U_i as the sum of these vector fields X_j's except X_i. See Figure 3 for the flow lines of X in the case where $i = 0$, *i.e.*, τ_{C_0} does not appear.

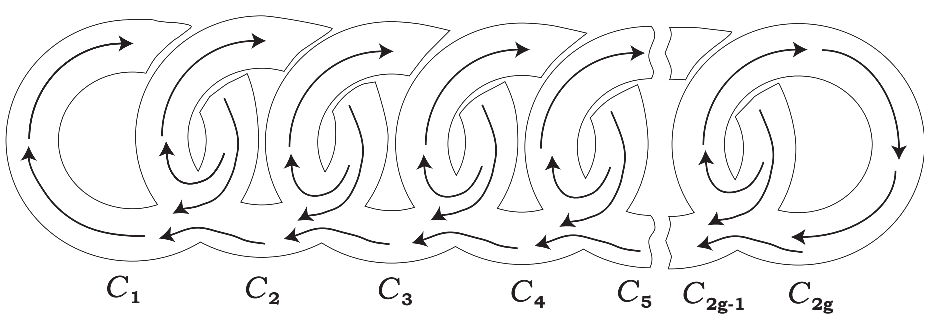

Figure 3: Flow generated by X_0

Since each component of $F - U_i$ contains at least one of the boundary component of F, the vector field X extends to a non-vanishing vector field on F. Temporarily, we put this vector field on each fibre $F \times \{\omega\}$ $(\omega \in [0, 1])$ and then adjust it in each small neighbourhood of $F \times \left\{\dfrac{k}{l}\right\}$ as follows: Let c_k denote the core curve $c_k = C_{i_k} \times \left\{\dfrac{k}{l}\right\}$ of the Dehn twist τ_k. Then by a homotopy of vector fields, we deform the vector field X on each fibre $F \times \left\{\dfrac{k}{l} + t\right\}$ $(-\epsilon < t < \epsilon$ for a sufficiently small $\epsilon > 0)$ so that the flow lines are parallel to c_k in a small neighbourhood $N(c_k)$ in the mapping torus. Note that we can deform the flow at each "crossroad" so that the resultant flow along one core of the two crossing annuli is prior to the other. Hence the deformation can be done. See Figure 4.

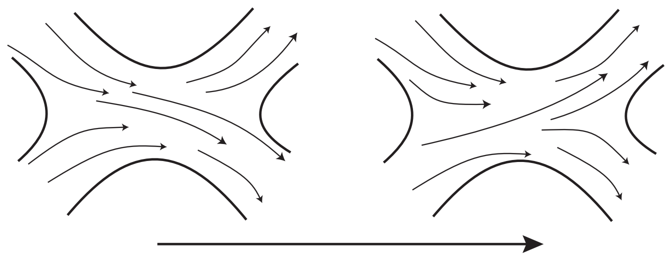

Figure 4: At "crossroads"

Thus we have a vector field at each leaf $F \times \left\{\dfrac{k}{l}\right\}$ which is invariant under the action of the Dehn twist τ_k. Consequently, we have a non-singular vector field on $M - \mathrm{int}N(L)$ tangent to each $F_\omega = F \times \{\omega\}$. Let ξ_0 denote this vector field.

Note that away from the support of the Dehn twists, especially near $\partial F \times S^1_\omega$, we can assume that the vector field ξ_0 commutes with $\dfrac{\partial}{\partial \omega}$. In fact, on a small neighbourhood V of the boundary $\partial(M - \mathrm{int}N(L)) = \{(\theta, 4, \omega)\}$ we may put $\xi_0|V = \sin(2\pi\chi\theta)\Theta + \cos(2\pi\chi\theta)\Omega$ where χ is the Euler characteristic of F.

As we will see soon below, we have to modify ξ_0 into ξ_1 in the following manner. Fix a Riemannian metric, so that each tangent plane $T_p\mathcal{F}_\varphi$ $(p \in M - \mathrm{int}N(L))$ admits an S^1-action which is nothing but the rotation of $T_p\mathcal{F}_\varphi$, respecting the orientation of each leaves. ξ_1 is obtained by rotating $\xi_0|F_\omega$ by $-2\pi\omega$ in the above sense, *i.e.*,

$$\xi_1|V = \sin 2\pi(\omega + \chi\theta)\Theta + \cos 2\pi(\omega + \chi\theta)\Omega.$$

Next let us consider a vector field ξ_2 on R tangent to $\mathcal{F}_\varphi|R$. Define $\xi_2|\partial R$ as

$$\xi_2|\partial R = -\sin 2\pi(\omega + \chi\theta)\frac{\partial}{\partial\theta} + \cos 2\pi(\omega + \chi\theta)\frac{\partial}{\partial\omega}.$$

Remark that $\Theta|\partial R = -\dfrac{\partial}{\partial\theta}$ and $\Omega|\partial R = \dfrac{\partial}{\partial\omega}$. Because $\xi_2|\partial R$ rotates minus once along the meridian $S^1_\omega \times \{\theta\} \times \{1\}$, it naturally extends to the whole of R as a non-singular vector field ξ_2 (see Figure 5).

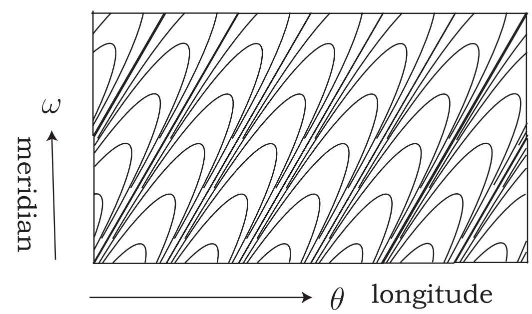

Figure 5: The flow induced on the border leaf for $\chi = -3$.

For example, taking a positive smooth function $g(r)$ on $(0,3)$ such that $2\pi r g(r) \equiv 1$ on $(0,1]$ and $g(r) \equiv 1$ on $[2,3)$, we may put

$$\xi_2|(R-L) = \sin 2\pi(\omega + \chi\theta)\Theta + g(r)\cos 2\pi(\omega + \chi\theta)\Omega.$$

Actually, setting $x = r\cos 2\pi\omega$ and $y = r\sin 2\pi\omega$ on the tubular neighbourhood $W = S^1_\theta \times D^2(1)$ of L of radius 1, we can rewrite ξ_2 on W as

$$\xi_2|W = \sin 2\pi\chi\theta\frac{\partial}{\partial x} + \cos 2\pi\chi\theta\frac{\partial}{\partial y} + f_1(r)\sin 2\pi(\omega + \chi\theta)\frac{\partial}{\partial\theta} \quad (\neq 0).$$

Finally we fill up $N(L) - \mathrm{int}R$ with $\xi_3 = \sin 2\pi(\omega+\chi\theta)\Theta + \cos 2\pi(\omega+\chi\theta)\Omega$. Apparently, the vector fields ξ_1, ξ_2 and ξ_3 match up to each other and define a non-singular vector field on whole of M which is tangent to $\mathcal{F}_\varphi$.

3.3 Computation of $e(T\mathcal{F})$

In order to make the arguments in the previous section clearer as well as to prepare for the following sections, let us consider the Poincaré dual $\mathrm{PD}[e(T\mathcal{F})] \in H_1(M;\mathbb{Z})$ to the Euler class. Morita studied this class from the view point of crossed homomorphism on the mapping class groups (Proposition 4.1 in [M1]. See also Proposition 5.3 in [M2] as well). Here we give it a more elementary description directly related to Dehn twist.

Fix a vector field X tangent to the foliation $\mathcal{F}$. Then, $\mathrm{PD}[e(T\mathcal{F})]$ is localized to a neighbourhood of the set $\mathcal{S}(X)$ of singular points of the vector field X.

For a simple closed oriented curve C on a leaf F_C with its tubular neighbourhood $N(C)$ in M, let $l \subset \partial N(C) \cap F_C$ and m be its *leaf longitude* and meridian. Then performing a Dehn surgery on C which attaches a new meridian disk along $j \cdot l - m$ ($j \in \mathbb{Z}$) is equivalent to cutting M along an annular neighbourhood $A(C)$ of C in F_C and pasting the downside A^- back to the upside A^+ (*w.r.t.* the transverse orientation of $\mathcal{F}$) by a diffeomorphism

$\psi : A^- \to A^+$, which is nothing but the j-th power τ_C^j of the right-handed Dehn twist along C on $A(C)$. Let M_ψ and $\mathcal{F}_\psi$ denote the resultant manifold and foliation respectively. Assume $\mathcal{S}(X) \cap \overline{N(C)} = \emptyset$. Then we can define the rotation number $\rho(C, X)$ of $X|C$ *w.r.t.* the tangent vector field $\dot{C}$ and regard the original Poincaré dual $\mathrm{PD}[e(T\mathcal{F})]$ as an element of the new homology $H_1(M_\psi; \mathbb{Z})$.

Lemma 1 *Under the above situation, we have*

$$\mathrm{PD}[e(T\mathcal{F}_\psi)] = \mathrm{PD}[e(T\mathcal{F})] - j\rho(C, X) \cdot [l] \in H_1(M_\psi; \mathbb{Z}).$$

Moreover, if $\rho(C, X) = 0$, the vector field X naturally induces a new vector field X_ψ on M_ψ without new singularities.

For the proof, see the next subsection. Now, let us explain the proof of Theorem a by this lemma. The construction of the vector field ξ_0 around the support of φ is explained by the latter half of the lemma. Then, in anyway, Lemma 1 implies that we can construct a vector field whose singular set is contained in a neighbourhood of the (connected) toral leaf ∂R. Then the meridian component of $\mathrm{PD}[e(T\mathcal{F}_\varphi)] \in H_1(N(L) - \mathrm{int}R; \mathbb{Z}) \approx \mathbb{Z}^2$ is killed by the meridian disk of R in $H_1(R; \mathbb{Z})$ and the longitudinal component is killed by F_ω in $H_1(M - \mathrm{int}N(L))$. We could choose any integer as χ in the construction of the vector field ξ_2 in the previous subsection. This fact corresponds to the first annihilation above. We could resolve the discordance of ξ_0 and ξ_3 by modifying ξ_0 into ξ_1. This corresponds to the second.

3.4 Proof of Proposition b

As is mentioned above, a mapping torus of a Dehn twist on a surface can be considered as the result of a Dehn surgery on the core curve of the Dehn twist in the surface times S^1. Thus we have another description, the surgery description of $(M, \mathcal{F}_\varphi)$. Suppose that M has a spinnable structure with the fibration $M - L \to S^1$ and its monodromy φ can be presented as $\varphi = \prod_{k=1}^{l} \tau_k^{j_k}$, where τ_k is the Dehn twist along C_{i_k}. Set

$$M' = (F \times S^1) \cup (S^1 \times D^2)$$

where the solid torus $S^1 \times D^2 \ni (\theta, r, \omega)$ is attached to the mapping torus of the identity $F \times [0,1]/\mathrm{id}_F = F \times S^1$ by the natural identification $S^1 \times \partial D^2 \approx S^1_\theta \times S^1_\omega \approx \partial F \times S^1$. The (non-twisted) product foliation $\{F \times \{\omega\}; \omega \in \mathbb{R}/\mathbb{Z}\}$ extends to the trivial spinnable foliation $\mathcal{F}_{\mathrm{id_F}}$. Then $(M, \mathcal{F}_\varphi)$ is the result of the Dehn surgeries on $C_{i_k} \times \left\{\frac{k}{l}\right\}$ ($k = 1, 2, \ldots, l$) which attach new meridian disks along $j_k \cdot l_k - m_k$ where l_k and m_k denote the leaf longitude and meridian of each C_{i_k} *w.r.t.* the leaf $F \times \left\{\frac{k}{l}\right\}$.

Now, we present (the Poincaré dual to) the Euler class $e(T\mathcal{F}_\varphi)$ for $(M,\ \mathcal{F}_\varphi)$. We orient the loops C_i ($i = 1, 2, \ldots, 2g$) as they are depicted in Figure 6.

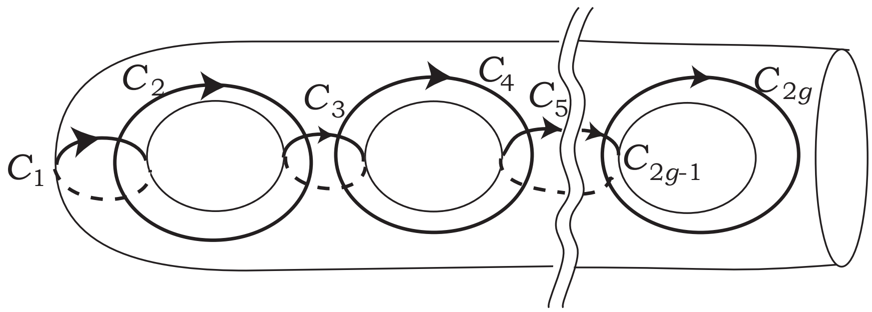

Figure 6: A basis for $H_1(F; \mathbb{Z})$.

Then we orient C_0 so that $[C_0] + [C_1] + [C_3] = 0$. For a non-vanishing vector field X tangent to the product foliation $\{F \times \{\omega\}\}$ on $F \times S^1$ and a smooth loop C on F, let $\rho(C, X)$ denote the rotation number of $X|C$ *w.r.t.* the tangent vector field $\dot{C}$.

Lemma 2 *Suppose that the monodromy* φ *is presented as* $\varphi = \prod_{k=1}^{l} \tau_k^{j_k}$ *where* $\tau_k = \tau_{C_{i_k}}$ ($i_k \in \{0, 1, \ldots, 2g\},\ j_k \in \mathbb{Z} - \{0\},\ k = 1, 2, \ldots, l$). *Then we have*

$$\mathrm{PD}[e(T\mathcal{F}_\varphi)] = -\sum_{k=1}^{l} j_k \cdot \rho(C_{i_k}, X) \cdot \left[C_{i_k} \times \left\{\frac{k}{l}\right\}\right] \in H_1(M; Z).$$

Proof. It is clear that $T\mathcal{F}_\varphi$ is trivial on $M - \bigcup_{k=1}^{l} N\left(C_{i_k} \times \left\{\frac{k}{l}\right\}\right)$. Let D_k be the meridian disk of the tubular neighbourhood of $C_{i_k} \times \left\{\frac{k}{l}\right\}$. Then the boundary curve ∂D_k goes around C_{i_k} direction j_i times. A non-vanishing vector field X on ∂D_k is a non-zero section induced from the outside of D_k and therefore $-j_k \cdot \rho(C_{i_k}, X)$ is the evaluation of $e(T\mathcal{F}_\varphi)$ at D_k. This implies Lemma 2.

Now we proceed to the proof of Proposition b. The monodromy φ is presented as

$$\varphi = \tau_{C_0}^{j_0}\tau_{C_1}^{j_1}\tau_{C_3}^{j_3} \cdot \prod_{k=4}^{l} \tau_k^{j_k} \quad (\tau_k = \tau_{C_{i_k}},\ i_k \in \{5, 6, \ldots, 2g\},\ k = 4, \ldots, l)$$

where τ_{C_2} and τ_{C_4} do not appear. Then let P denote the subsurface of F which is bounded positively by $C_0 \cup C_1 \cup C_3$. P is homeomorphic to a 3-punctured sphere and inherits the orientation from F. We will define three tori T_{01}, T_{13} and T_{30} in M' as follows. Let t_{01} be a loop in $F \subset M'$ which intersects with C_0 and C_1 once respectively and does not intersect with the other curves C_i $(i \neq 0, 1)$. See Figure 7.

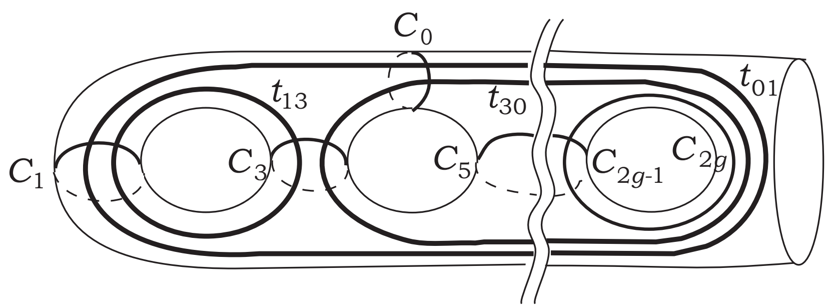

Figure 7: The section of the tori at F.

Then rotating t_{01} around L over the base S^1_ω, we have a torus T_{01} in M' with $T_{01} \cap F = t_{01}$. Similarly, we have other two tori T_{13} and T_{30} in M'. We give an orientation on T_{pq} so that the normal orientation of T_{pq} coincides with the orientation of C_q (see Figure 7).

On the surgery description, we can assume that the Dehn surgery on C_0, C_1 and C_3 are performed on the same level $F \times \{0\}$. Remove the interior of the tubular neighbourhoods of C_0, C_1 and C_3 in M' and denote the resultant 3-manifold by M''. By the abuse of language, we also denote $T_{ij} \cap M''$ by T_{ij}. Take n copies of P, m_{01} copies of T_{01}, m_{13} copies of T_{13} and m_{30} copies of T_{30} and perform a double curve surgery on them to obtain a surface in M''. See Figure 8. Here n and m_{pq}'s are integers and their signs mean the orientations of the surfaces.

Then if the following equation has an integral solution for some non-zero integer n, then the boundary of the resultant surface can be capped with meridian disks of the tubular neighbourhoods of C_0, C_1 and C_3 in M:

$$\begin{pmatrix} n \\ n \\ n \end{pmatrix} = \begin{pmatrix} 0 & j_0 & -j_0 \\ -j_1 & 0 & j_1 \\ j_3 & -j_3 & 0 \end{pmatrix} \begin{pmatrix} m_{13} \\ m_{30} \\ m_{01} \end{pmatrix}. \tag{1}$$

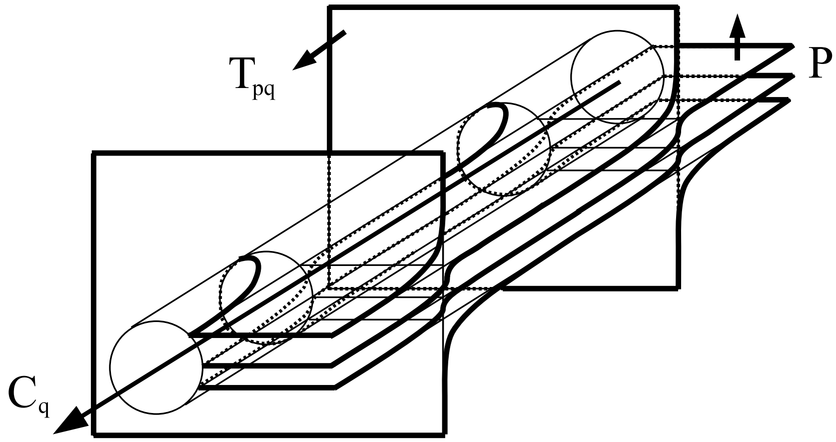

Figure 8: The double curve surgery.

Simple calculation shows that the equation (1) has an integral solution if and only if $\frac{1}{j_0}+\frac{1}{j_1}+\frac{1}{j_3}=0$ and n is a common multiple of j_0, j_1, and j_3. Hence the condition (ii) assures the existence of such a closed surface in M that algebraically intersects with C_0, C_1 and C_3. This implies that the Euler class $e(T\mathcal{F}_\varphi)$ is of infinite order. Especially if we take n to be $\tilde{n}=\mathrm{lcm}(j_0,\, j_1)$ $(=\mathrm{lcm}(j_1,\, j_3)=\mathrm{lcm}(j_3,\, j_0)>0)$, the surface is connected.

To show the converse, we first calculate the Euler class precisely. We can choose the vector field X so that the Dehn twist τ_{C_i} preserves X for $i\geq 5$. Then $\rho(C_i,\, X)=0$ for $i\geq 5$ and therefore

$$\mathrm{PD}[e(T\mathcal{F}_\varphi)]=-(j_0\rho_0[C_0]+j_1\rho_1[C_1]+j_3\rho_3[C_3]) \quad (\rho_i=\rho(C_i,\, X), i=0,1,3).$$

Moreover it is easy to see that we can assume $\rho_1=1(=-\chi(P))$, $\rho_0=\rho_3=0$ by choosing a suitable vector field X. Consequently we have $\mathrm{PD}[e(T\mathcal{F}_\varphi)]=-j_1[C_1]$. Let J denote the sum $j_1j_3+j_3j_0+j_0j_1$. Clearly, $\frac{1}{j_0}+\frac{1}{j_1}+\frac{1}{j_3}=0$ if and only if $J=0$ and $j_0j_1j_3\neq 0$. Thus the following lemma implies the converse.

Lemma 3 *The homology class $J\cdot\mathrm{PD}[e(T\mathcal{F}_\varphi)]$ vanishes in $H_1(M;\mathbb{Z})$.*

Proof. Consider the following exact sequence:

$$H_1(F;\mathbb{Z})\xrightarrow{\varphi_*-\mathrm{id}} H_1(F;\mathbb{Z})\to H_1(M;\mathbb{Z}).$$

We take $[C_i]$, $i=1,\,2,\,3,\,5,\,\ldots,\,2g$ and $[\widehat{C}_4]$ as another basis for $H_1(F;\mathbb{Z})$, where $\widehat{C}_4$ is the oriented loop depicted in Figure 9.

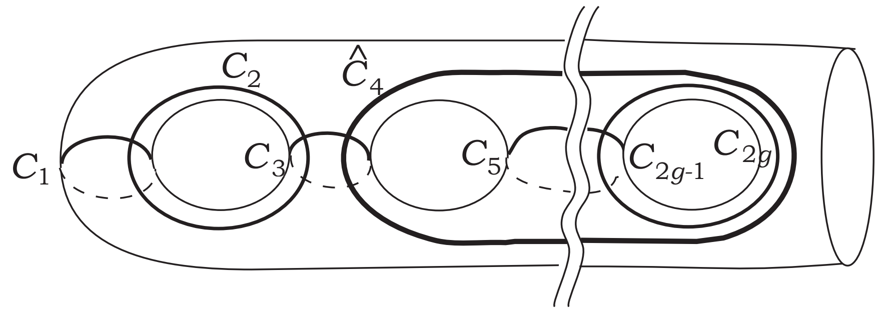

Figure 9: Another basis for $H_1(F;\mathbb{Z})$.

Note that $\widehat{C}_4$ is isotopic to T_{30} and $[\widehat{C}_4] = \sum_{i=2}^{g}[C_{2i}]$. Let G denote the subgroup of $H_1(F;\mathbb{Z})$ generated by $[C_1]$, $[C_2]$, $[C_3]$ and $[\widehat{C}_4]$. Then since neither of τ_{C_2} nor τ_{C_4} appears in φ, G is invariant by φ_*. We restrict φ_* to G. Then $\varphi_*|G$ is represented as

$$\varphi_*|G = \begin{pmatrix} 1 & j_1 & 0 & j_0 \\ 0 & 1 & 0 & 0 \\ 0 & -j_3 & 1 & j_0 + j_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\mathrm{PD}[e(T\mathcal{F}_\varphi)] = -j_1[C_1]$ and the exactness of the above sequence, it is sufficient to show that there exists $(x_1,\, x_2,\, x_3,\, x_4) \in \mathbb{Z}^4$ with

$$J \cdot \mathrm{PD}[e(T\mathcal{F}_\varphi)] = J \begin{pmatrix} -j_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & j_1 & 0 & j_0 \\ 0 & 0 & 0 & 0 \\ 0 & -j_3 & 0 & j_0 + j_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

that is, to show that there exists $(x_2,\, x_4) \in \mathbb{Z}^2$ with

$$-j_1 J \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} j_1 & j_0 \\ -j_3 & j_0 + j_3 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}.$$

Now the determinant of the matrix $A = \begin{pmatrix} j_1 & j_0 \\ -j_3 & j_0 + j_3 \end{pmatrix}$ is just J. Therefore if $J \neq 0$, then the inverse matrix A^{-1} can be written as $\dfrac{1}{J}\tilde{A}$, where $\tilde{A}$ is a 2×2 integer matrix. Hence $-j_1\tilde{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the desired integer vector. This implies Lemma 3.

Consequently the proof of Proposition b is completed.

3.5 Proof of Theorem c

Under the hypothesis of the theorem, Proposition b implies $\dfrac{1}{j_0} + \dfrac{1}{j_1} + \dfrac{1}{j_3} = 0$. Moreover, we can construct a closed surface whose evaluation with $e(T\mathcal{F}_\varphi)$ is non-zero. Suppose that such a closed surface S is constructed with $\tilde{n}$ copies of P, m_{13} copies of T_{13}, m_{30} copies of T_{30} and m_{01} copies of T_{01} as in the proof of Proposition b. Then after performing double curve surgery on $(\cup_{\tilde{n}} P) \cup (\cup_{m_{13}} T_{13}) \cup (\cup_{m_{30}} T_{30}) \cup (\cup_{m_{01}} T_{01})$, the number of the boundary components of the resultant surface b is equal to the sum $\dfrac{\tilde{n}}{|j_0|} + \dfrac{\tilde{n}}{|j_1|} + \dfrac{\tilde{n}}{|j_3|}$, because the equation $\tilde{n} = j_0 m_{30} - j_0 m_{01}$, for example, implies the resultant curve on the boundary component corresponding to C_0 is $\dfrac{\tilde{n}}{j_0} = m_{30} - m_{01}$ copies of the loop of the slope j_0. Then the Euler characteristic $\chi(S)$ is calculated as follows.

Claim 1. $\chi(S) = -2m + \tilde{n}\chi(P) + b$, where $m = |m_{13}| + |m_{30}| + |m_{01}|$.

Proof. The Euler characteristic $\chi(S)$ is equal to that of the disjoint union of the material surfaces, *i.e.*, $|m_{13}| + |m_{30}| + |m_{01}|$ copies of a twice punctured torus, $\tilde{n}$ copies of a 3-punctured sphere, and b copies of a disk. This implies Claim 1.

On the other hand, the Euler number of $T\mathcal{F}_\varphi|S$ is calculated as follows.

Claim 2. $\langle e(T\mathcal{F}_\varphi), [S]\rangle = -\tilde{n} (= \tilde{n}\chi(P))$.

Proof. Since the Euler class can be written as $\mathrm{PD}[e(T\mathcal{F}_\varphi)] = -j_1[C_1]$, we have

$$\begin{aligned}
\langle e(T\mathcal{F}_\varphi), [S]\rangle &= \langle -j_1[C_1], [S]\rangle \\
&= -j_1(m_{01} - m_{13}) \\
&= -j_1\left(\frac{\tilde{n}}{j_1}\right) \\
&= -\tilde{n},
\end{aligned}$$

as is desired.

Now, by Claims 1 and 2, the equality $|\langle e(T\mathcal{F}_\varphi), [S]\rangle| = -\chi(S)$ holds if and only if $b = 2m$. As is noted above, the number b of the boundary components of S is equal to the sum $\dfrac{\tilde{n}}{|j_0|} + \dfrac{\tilde{n}}{|j_1|} + \dfrac{\tilde{n}}{|j_3|}$ and is independent from m_{pq}, m_{qr} and m_{rp}.

Claim 3. The inequality $b \leq 2m$ holds where the equality $b = 2m$ actually holds for the minimum possible m.

Proof. Since $\dfrac{1}{j_0} + \dfrac{1}{j_1} + \dfrac{1}{j_3} = 0$, there is an even permutation (p, q, r) of $(0, 1, 3)$ such that $\dfrac{1}{|j_r|} = \dfrac{1}{|j_p|} + \dfrac{1}{|j_q|}$. For the equation (1) with $n = \tilde{n}$, we have the general solution

$$(m_{pq},\, m_{qr},\, m_{rp}) = \left(k,\, -\frac{\tilde{n}}{j_q} + k,\, \frac{\tilde{n}}{j_p} + k\right)$$

where k is any integer. Then we have

$$\begin{aligned} 2m &= 2\left(|k| + \left|-\frac{\tilde{n}}{j_q} + k\right| + \left|\frac{\tilde{n}}{j_p} + k\right|\right) \\ &\geq 2\left(\frac{\tilde{n}}{|j_q|} + \frac{\tilde{n}}{|j_p|}\right) \\ &= \tilde{n}\left(\frac{1}{|j_q|} + \frac{1}{|j_p|} + \frac{1}{|j_r|}\right) \\ &= b \end{aligned}$$

where the equality holds when $k = 0$. This implies Claim 3.

Claim 3 implies that $\langle e(T\mathcal{F}_\varphi), [S]\rangle = -\chi(S)$ holds for the minimum possible m. On $F = F \times \{0\} \subset M_\varphi$, we can find an arc a which connects a point of t_{01} and a point of ∂F and $\operatorname{int} a$ does not intersect with C_i's nor t_{pq}'s. See Figure 10.

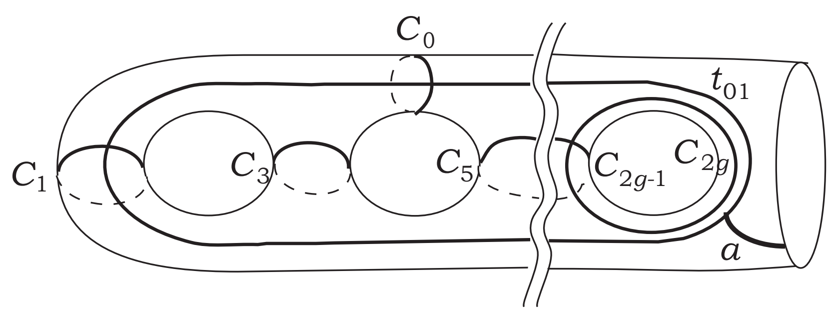

Figure 10: The arc connecting t_{01} and ∂F.

Then, rotating a around the base S^1_ω, we have an annulus A in M such that $\partial A = (A \cap T_{01}) \sqcup (A \cap \partial M_\varphi)$. By the construction, the loop $A \cap \partial M_\varphi$ is the meridian loop ∂D^2 of $N(L)$ so that the surface $A \cup D^2$ is a compressing disk of the surface S. Thus we can perform a surgery on S along this compressing disk to obtain a new closed connected surface S'. Then we have $\chi(S) < \chi(S') \leq 0$ and $[S] = [S'] \in H_2(M;\mathbb{Z})$. Finally, we have

$$\begin{aligned} \langle e(T\mathcal{F}_\varphi),\, [S']\rangle &= \langle e(T\mathcal{F}_\varphi),\, [S]\rangle \\ &= -\chi(S) \\ &> -\chi(S')\,. \end{aligned}$$

Thus the absolute inequality does not hold. This completes the proof of Theorem c.

4 The Relative Inequality

If relative Thurston-Bennequin's inequality for a contact structure holds, then so does the absolute one. As to Thurston's inequality for foliations, it is not the case in general.

In our situation, the violation of the absolute inequality in fact implies that of the relative one. In this section this is explained in §4.2 in a direct and geometric manner and also in §4.1 by more general argument for spinnable foliations. It yields an application on the mapping class of the monodromy, which is given in the final subsection.

4.1 Bennequin's Lemma

Proposition 4 *For spinnable foliations relative Thurston's inequality is stronger than the absolute one.*

In this subsection we explain this Proposition. The basic idea is to pass from foliations to contact structures. Let $\{\xi_i\}$ be a sequence of contact structures converging to a foliation $\mathcal{F}$ as oriented plane fields. Therefore ξ_i's are isomorphic to $T\mathcal{F}$ as an oriented plane bundle, so that their Euler classes coincide. Therefore absolute Thurston's inequality holds if and only if absolute Thurston-Bennequin's inequality holds for ξ_i's.

On the other hand, as to relative inequalities, the situation is more delicate. If ξ_i's satisfy relative Thurston-Bennequin's inequality, relative Thurston's inequality holds for $\mathcal{F}$ as well, because any transverse knot to $\mathcal{F}$ is also transverse to ξ_i's for large i's. However, even if we take i large enough, it is not true in general that transverse knot to ξ_i is transverse to $\mathcal{F}$, and in fact, there exists a sequence of over twisted contact structures converging to a foliation which satisfies relative Thurston's inequality. For spinnable foliations, the situation is better. First, we know a good family of contact structures for a spinnable foliations.

Lemma 5([Mo]) *Thurston-Winkelnkemper's contact structure associated with a spinnable structure $\mathcal{S}_\varphi$ has an isotopic family which converges to the spinnable foliation $\mathcal{F}_\varphi$.*

Proof. Recall that Thurston-Winkelnkemper's contact structure for the spinnable structure on $M = N(L) \cup F \times [0,1]/\varphi$ is the kernel of a contact 1-form α_1 with

$$d\alpha_1 | \mathrm{int}\, F \times \{\omega\} > 0 \ \ (\forall \omega \in \mathbb{R}/\mathbb{Z}) \quad \text{and} \quad \alpha_1 | N(L) = d\theta + (r^2/4) d\omega$$

where (θ, r, ω) is the coordinate on $N(L) \approx S^1_\theta \times D^2(4)$ described in §3. Then, for the Pfaff form α_0 in §3, we see that the family $\{(1-t)\alpha_0 + t\alpha_1;\, t \in (0,1]\}$ of contact forms defines an isotopic family of contact structures from Gray's stability theorem. This family of contact structures actually converges to the spinnable foliation.

Here is the key step of this argument.

Bennequin's Lemma *Any positive transverse link of Thurston-Winkelnkemper's contact structure associated with a given spinnable structure $\mathcal{S}_\varphi$ is isotopic through a family of transverse links to one in the mapping torus $F \times [0,1]/\varphi$ which is positively transverse to each fibre $F \times \{\omega\}$ $(\omega \in \mathbb{R}/\mathbb{Z})$.*

Now, let us assume that $\mathcal{F}$ satisfies the relative inequality. Then, ξ_i's also satisfy the relative inequality and hence the absolute one as well. This implies $\mathcal{F}$ satisfies absolute Thurston's inequality.

Bennequin's Lemma is proved essentially in the same way as in his original work (the proof of Theorem 8 in [B]). The detail will be given in a forthcoming paper.

Remark. Let us take a Reeb component and another copy of it with up side down. They are glued together along the toral boundary to be a foliation $\mathcal{F}$ on $S^2 \times S^1$. It is easy to see that there is no Seifert surface with positive transverse boundary and therefore the relative inequality holds, while absolute one does not hold since $\mathrm{PD}(e(T\mathcal{F})) = \pm 2[S^1]$. The same situation also happens for the product foliation $\mathcal{F}_\zeta = \{S^2 \times *\}$ on $S^2 \times S^1$. We know of essentially no other such examples and believe that in fact there are very few. Practically it seems not hard to deduce the absolute inequality from the relative one in each individual cases.

4.2 Closed Surface S and Seifert Surface Σ

In order to construct a Seifert surface Σ which violates the relative inequality, we first describe the closed surface S more precisely in the proof of Theorem c. Take an even permutation $(p,\, q,\, r)$ of $(0,\, 1,\, 3)$ such that $\dfrac{1}{|j_r|} = \dfrac{1}{|j_p|} + \dfrac{1}{|j_q|}$ as in the proof of Claim 3. Then the special solution

$$(m_{pq},\, m_{qr},\, m_{rp}) = \left(0,\, -\frac{\tilde{n}}{j_q},\, \frac{\tilde{n}}{j_p}\right)$$

for the equation (1) roughly determines the surface S.

On $P \times \mathbb{R}/\mathbb{Z} \subset M$, we give S a detailed description as follows. Let us regard P as a branched double covering over an annulus $A = \mathbb{R}/\mathbb{Z} \times [0,1]$ with the branch point $(0, 1/2) \in A$ and $\mathcal{H}$ be the singular foliation on P, which is given as the pull-back of the simple foliation on A defined by the first projection $A \to \mathbb{R}/\mathbb{Z}$. Let $\varpi : P \to \mathbb{R}/\mathbb{Z}$ denote this projection composed with the branched covering. They are depicted in Figure 11, where the outer boundary of P presents C_r and the left and the right inner boundaries present C_q and C_p respectively. The surface P is divided into two pieces P_q and P_p as in the figure and $\varpi_i : P_i \to \mathbb{R}/\mathbb{Z} \approx C_i$ denotes the restriction of ϖ to P_i $(i = q, p)$.

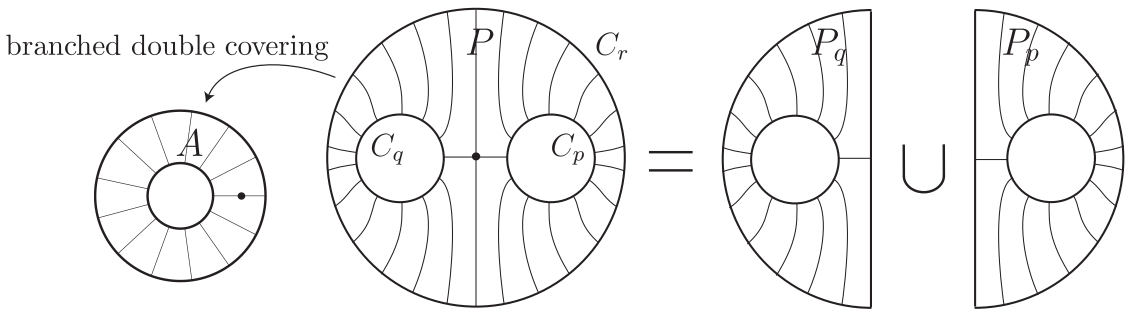

Figure 11: The singular foliation $\mathcal{H}$ on $P = P_q \cup P_p$.

Then we define the surface S by setting for each $\omega \in \mathbb{R}/\mathbb{Z}$

$$S \cap (P \times \{\omega\}) = \left(\bigcup_{u \in \mu_q(\omega)} \varpi_q^{-1}(u) \right) \cup \left(\bigcup_{v \in \mu_p(\omega)} \varpi_p^{-1}(-v) \right) \times \{\omega\}$$

where $\mu_i(\omega)$ denotes the set $\{u \in \mathbb{R}/\mathbb{Z}; u \equiv j_i\omega (\operatorname{mod} j_i/\tilde{n})\}$ $(i = q, p)$. Then $S \cap (P \times \mathbb{R}/\mathbb{Z})$ has $\tilde{n}$ positive hyperbolic tangent points to the foliation $\mathcal{F}_\varphi$. The surface S coincides with $(\cup_{m_{qr}} T_{qr}) \cup (\cup_{m_{rp}} T_{rp})$ outside $P \times \mathbb{R}/\mathbb{Z}$ and has no other tangencies. Therefore we see $\chi(S) = -\tilde{n}$ and $\langle e(T\mathcal{F}_\varphi), [S] \rangle = -\tilde{n}$ by summing up the indices or the signed indices of tangencies respectively. Figure 12 shows the surface $S \cap (P \times \mathbb{R}/\mathbb{Z})$ for $(j_0, j_1, j_3) = (-3, -6, 2)$ and $\tilde{n} = \operatorname{lcm}(-3, -6) = 6$. Here $(m_{01}, m_{13}, m_{30}) = (0, 1, -2)$ determines S. The closed oriented surface S has six positive hyperbolic tangent points to $\mathcal{F}_\varphi$ and we see

$$\chi(S) = -6 \quad (\operatorname{genus}(S) = 4) \quad \text{and} \quad \langle e(T\mathcal{F}_\varphi), [S] \rangle = -6.$$

Now we modify the surface S into a Seifert surface Σ by using compression disks which are transverse to the axis of $\mathcal{S}_\varphi$. Like the arc a on F used in the proof of Theorem c, which connects a point on the boundary ∂F and a point of the subset $P \cup t_{qr} \cup t_{rp} \cup t_{pq} (\subset F)$

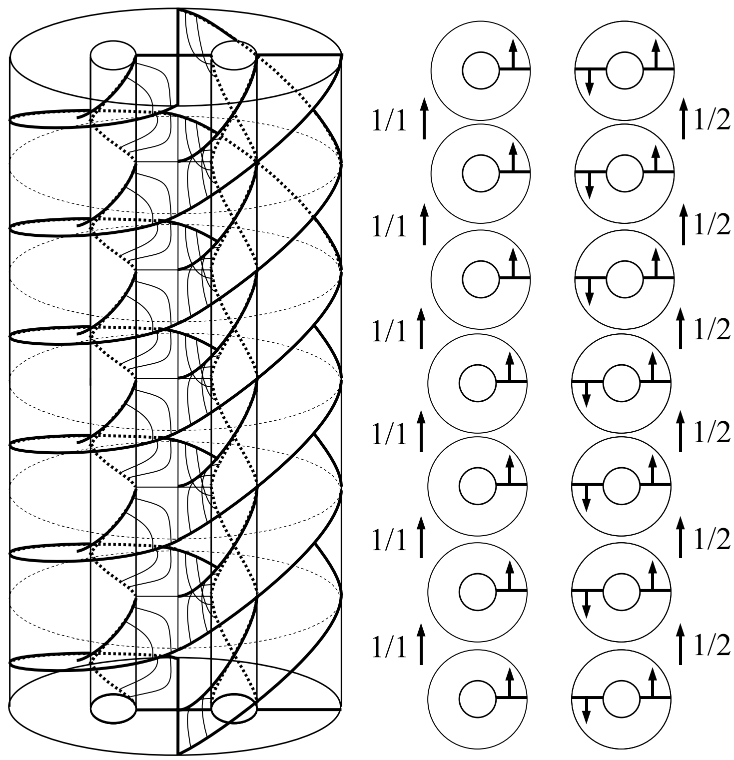

Figure 12: The surface S.

without touching the support of φ, there also exist a mutually disjoint pair of similar arcs a_1 and a_2 with terminal points on t_{qr} and t_{rp} respectively. Then the arcs a_1 and a_2 trace collars of two compression disks D_1 and D_2 of S on M. Perform a surgery on S along D_1 and D_2 disks to obtain a closed connected surface S''. The surface S'' contains two copies with opposite orientations of each of D_1 and D_2. Therefore it has two positive elliptic tangent points and two negative elliptic ones other than $\tilde{n}$ positive hyperbolic tangencies. Choose the negative copy of D_1 in S'' and remove its interior from S''. Then, we obtain a Seifert surface Σ. §2.1 and especially Figure 1 tells its boundary $L = \partial\Sigma = -\partial(\text{removed disc})$ is positively transverse to $\mathcal{F}_\varphi$. Then we see that $\chi(\Sigma) = 3 - \tilde{n}$ and $-\mathrm{lk}(L, L^X) = 1 - \tilde{n}$ by summing up the indices and the signed indices respectively. Thus the Seifert surface Σ violates $\mathrm{lk}(L, L^X) \leq -\chi(\Sigma)$.

4.3 A Result on Mapping Classes

The violation of relative Thurston's inequality for the above spinnable structure $\mathcal{F}_\varphi$ implies the overtwistedness of Thurston-Winkelnkemper's contact structure associated with the same spinnable structure $\mathcal{S}_\varphi$. Moreover, as is mentioned in §1, this implies that the monodromy φ is never isotopic to a product of only right-handed Dehn twists. Note that the conditions (i) and (ii) for $\mathcal{S}_\varphi$ are also satisfied for the spinnable structure $\mathcal{S}_{\varphi^{-1}}$ on $-M = N(L) \cup F \times [0,1]/\varphi^{-1}$. Thus relative Thurston's inequality does not hold for the spinnable foliation $\mathcal{F}_{\varphi^{-1}}$ on $-M$, which is different from $\mathcal{G}_\varphi$ described in Remark in §2.1. This implies that the inverse diffeomorphism φ^{-1} is not isotopic to a product of right-handed Dehn-twists. Therefore the original monodromy φ is not isotopic to a product of left-handed Dehn twists. Thus we obtain a result on the mapping classes:

Corollary e *If a spinnable structure $\mathcal{S}_\varphi$ satisfies the conditions* (i) *and* (ii)*, the monodromy map φ is isotopic neither to a product of right-handed Dehn-twists nor to a product of left-handed Dehn-twists.*

Note that if we compose φ with a large power of right-handed (resp. left-handed) Dehn twist along a parallel loop of ∂F then we obtain a diffeomorphism isotopic to a product of right-handed (resp. left-handed) Dehn-twists. The existence of the arc a in the proof of Theorem c or in the condition (4) of (II) in §2 forbids adding extra twists along a parallel loop of ∂F.

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