A BIRKHOFF SECTION FOR THE BONATTI-LANGEVIN EXAMPLE OF ANOSOV FLOW

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Abstract. Bonatti and Langevin gave an example of Anosov flow which is transitive, admits a transverse torus, but is not topologically conjugate to a suspension flow. We give another description of this example as a suspension flow of a pseudo-Anosov map on the 2-sphere surgered along two closed orbits and generalize it to produce infinitely many examples with same properties.

1. Introduction

The suspensions of hyperbolic automorphisms of the 2-torus and the geodesic flows of hyperbolic surfaces are typical examples of Anosov flow on 3-manifolds. These examples are transitive, that is, they contain a dense orbit.

Verjovsky proved in [V] that in dimension equal to or greater than 4 every Anosov flow with a codimension one invariant foliation is transitive. This does not hold in dimension 3 and in fact Franks and Williams constructed non-transitive examples by using DA-operations [FW].

If an Anosov flow is not transitive then there exists a finite collection of tori transverse to the flow and it isolates the basic sets of the flow. It is natural to ask whether transitive Anosov flows with a transverse torus are always topologically conjugate to suspensions.

But the answer to this question turned out to be negative, because Bonatti and Langevin constructed an example of transitive Anosov flow on a graph manifold which admits a transverse torus but is not topologically conjugate to a suspension. Such examples can be found also in [Br2].

In [Ba], Barbot developed surgery techniques introduced in [HT] and [G] to prove that most of such surgeries on the transverse torus of the Bonatti-Langevin example can yield similar examples on graph manifolds, which he called *BL-flows*. Furthermore, he classified Anosov flows on such manifolds.

In this paper, we take another approach to the Bonatti-Langevin example. An orientable immersed surface Σ is called a *Birkhoff section* for a flow ϕ^t if it satisfies the following:

(1) the interior IntΣ is an embedded surface transverse to ϕ^t and the boundary $\partial\Sigma$ consists of closed orbits of ϕ^t,
(2) there is a $t_0 > 0$ such that every orbit meets Σ in any time interval of length t_0.

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In [F], Fried has proved that every transitive Anosov flow admits a Birkhoff section.

Let ϕ^t be a transitive Anosov flow and Σ a Birkhoff section for ϕ^t, which is a compact surface with non-empty boundary. We define Σ^* to be a closed surface obtained from Σ by collapsing each boundary component to a point. Then the flow ϕ^t naturally induces a first return map $f^*\colon \Sigma^* \to \Sigma^*$. The map f^* satisfies Anosov properties except for a finite number of points $p_1, \ldots, p_k \in \Sigma^*$ and the restrictions of the invariant foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ to Σ^* induce singular foliations F^u and F^s respectively. Such a map is called a *pseudo-Anosov map*. Conversely, the given Anosov flow ϕ^t can be retrieved from the suspension flow of this pseudo-Anosov map $f^*\colon \Sigma^* \to \Sigma^*$ by Dehn surgeries on the closed orbits corresponding to the singular points.

Since the Bonatti-Langevin example is transitive, it has a Birkhoff section. We describe a specific one and determine the first return map:

Theorem 1.1. *The Bonatti-Langevin example admits a Birkhoff section of genus 0 and with 4 singularities and the pseudo-Anosov map defined by the first return map has a branched covering by a hyperbolic automorphism of the 2-torus, which is represented by* $\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$

On the other hand, the Bonatti-Langevin example admits a transverse torus which is not a global section. We further investigate the relation between the Birkhoff section and the transverse torus to produce an infinite family of examples with the same properties.

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2. The Bonatti-Langevin example

In this section, we review the Bonatti-Langevin example of Anosov flow. For the precise description, see [BL].

We first construct the underlying 3-manifold M. In the following, S^1 is considered as a circle $\mathbb{R}/4\mathbb{Z}$. Define $\bar{M}$, a manifold with boundary, as $\bar{M} = \left(\mathbb{R} \times [-1,1] \setminus \bigcup_{i\in\mathbb{Z}} D((2i,0), \frac{1}{4})\right) \times S^1$ with coordinates (x, y, θ). Let $M_0 = \bar{M}/\langle\Phi\rangle$ be the quotient space of $\bar{M}$ by the action generated by $\Phi(x, y, \theta) = (x + 2, -y, -\theta)$. Then M_0 is topologically a non-trivial S^1-bundle over projective plane with two holes. Let T_1 be the boundary component corresponding to $\{|y| = 1\}$ and T_2 be the one corresponding to $\{x^2 + y^2 = \frac{1}{16}\}$. Consider coordinates $[x, \theta] = (x, 1, \theta)$ on T_1 and $[\omega, \theta] = (\frac{1}{4} \sin \frac{\omega}{2}\pi, \frac{1}{4} \cos \frac{\omega}{2}\pi, \theta)$ on T_2, where $[x, \theta], [\omega, \theta] \in S^1 \times S^1$. We define the gluing map $A\colon T_1 \to T_2$ by $A[x, \theta] = [\theta, -x]$ and then let $M = M_0/A$ be the quotient manifold by this map, where the glued torus is denoted by T.

Secondly, we define the vector field $Z = X + Y$ on M. We take $M_1 = \bar{M} \cap \{0 \le x \le 2\}$ as a fundamental domain of M. The vector field X is defined to have no $\frac{\partial}{\partial\theta}$-component and satisfy the following conditions (see Figure 1):

(1) $X = (x-1)\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ on $\{x = 1\} \cup \{y = 0\}$ and on a small neighborhood U of $\{x = 1, y = 0\}$;
(2) $X = -y\frac{\partial}{\partial y}$ on $\{x = 0\} \cup \{x = 2\}$;

(3) X is orthogonal to T_1 and T_2 and points inward on T_1 and outward on T_2;

(4) the orbit of X starting from $[x,\theta] \in T_1$ arrives at $[x,\theta] \in T_2$ for $-1 < x < 1$ and at $[4-x,-\theta] \in T_2$ for $1 < x < 3$.

The $\frac{\partial}{\partial\theta}$-component $Y = \beta(x,y)\frac{\partial}{\partial\theta}$ satisfies the following:

(1) $\beta(x,y) = 0$ on ∂M_1 and $\beta(x,y) > 0$ on $\mathrm{Int}M_1$;

(2) there exists a sufficiently large positive number $C_1 > 0$ such that $\frac{\partial\beta}{\partial x}(0,y) = -\frac{\partial\beta}{\partial x}(2,-y) \geq C_1$ holds for $-\frac{2}{3} \leq y \leq -\frac{1}{2}$ and $\frac{1}{2} \leq y \leq \frac{2}{3}$.

Assume further that $|Y| \gg |X|$ except for a small neighborhood of ∂M_1. Then we can take Z to be a smooth vector field on M which induces a smooth non-singular flow ϕ^t. This flow has a closed orbit γ in $\{1\} \times \{0\} \times S^1$, the only orbit that does not intersect the torus T. We also denote

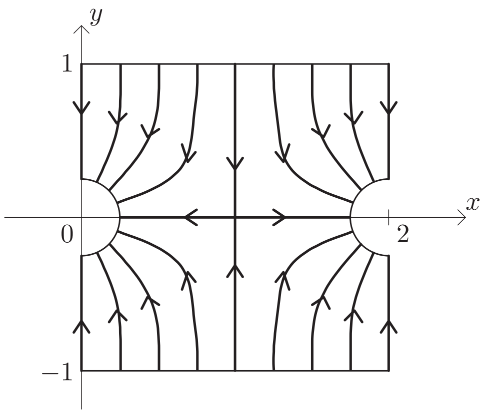

FIGURE 1. Vector field X

by c_0, c_1, and c_2 the closed orbits in $\{0\} \times [\frac{1}{4},1] \times \{2\} \cup \{2\} \times [\frac{1}{4},1] \times \{0\}$, $\{0\} \times [\frac{1}{4},1] \times \{0\}$, and $\{2\} \times [\frac{1}{4},1] \times \{2\}$, respectively.

Theorem 2.1 ([BL]). *The flow ϕ^t is a volume-preserving Anosov flow which is not conjugate to a suspension flow.*

Instead of proving this theorem, we compute the holonomy along c_1, which illustrates Anosov property. Let $[x,\theta]$, $[\omega,\theta]$ be the coordinates on T_1 and T_2 as above. The fiber at $(0,1,0)$ of the quotient bundle $TM/T\phi$ is naturally identified with $T_{[0,0]}T_1$ and $T_{[0,0]}T_2$. Then the linear holonomy along c_1 is represented by the composition of $D\phi^t\colon T_{[0,0]}T_1 \to T_{[0,0]}T_2$ and $DA^{-1}\colon T_{[0,0]}T_2 \to T_{[0,0]}T_1$, that is,

$$DA^{-1} \circ D\phi^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_2 & 1 \end{pmatrix} = \begin{pmatrix} -C_2 & -1 \\ 1 & 0 \end{pmatrix},$$

where C_2 is a sufficiently large positive number depending on C_1. Then the trace of this linear map is equal to $-C_2$, which is < -2 for adequate choice of C_1, and therefore this map is hyperbolic.

Remark 2.2. Since the trace is negative, the stable and unstable foliations of this Anosov flow are non-orientable and their leaves containing the closed orbit c_1 are homeomorphic to the Möbius band.

Remark 2.3. If we replace the gluing map $A\colon T_1 \to T_2$ with other diffeomorphisms, we obtain a family of different manifolds with a flow by Z. In [Ba], Barbot called them *BL-manifolds* and *BL-flows*. He further proved that if a BL-manifold is not a circle bundle, we can take BL-flow to be Anosov. By construction, Anosov BL-flows admit a transverse torus and contain a unique closed orbit which does not meet the transverse torus.

Remark 2.4. In the construction of vector field, replacing the function β by a greater one also induces an Anosov flow, which is known to be topologically conjugate to the original one by the classification theorem in [Ba].

3. Birkhoff section and the return map

In this section, we construct a Birkhoff section for ϕ^t in an explicit way. We continue to consider the coordinates in the fundamental domain M_1 defined in the previous section.

We define pieces of surface:

$$\begin{aligned} S_1 &= \{(x, y, -x+2) : 0 \le x \le 2,\ -1 \le y \le -\frac{1}{4}\} \cup \{(x, y, x) : 0 \le x \le 2,\ \frac{1}{4} \le y \le 1\} \\ &\quad \cup \{\big(x, \frac{1}{4}\cos\frac{\omega}{2}\pi, \frac{1-\omega}{1-\frac{1}{4}\sin\frac{\omega}{2}\pi}(x-1)+1\big) : \frac{1}{4}\sin\frac{\omega}{2}\pi \le x \le 2 - \frac{1}{4}\sin\frac{\omega}{2}\pi,\ 0 \le \omega \le 2\}, \\ S_2 &= \{(x, 1, \theta) : |x| + |\theta - 2| \le 2,\ 0 \le x \le 2\} \cup \{(x, -1, \theta) : |x-2| + |\theta - 2| \le 2,\ 0 \le x \le 2\}. \end{aligned}$$

and the edges in the boundary of them:

$$e_1 = \{(x, 1, x) : 0 \le x \le 2\}, \quad e_2 = \{(x, 1, -x+4) : 0 \le x \le 2\},$$
$$e_3 = \{(x, -1, -x+2) : 0 \le x \le 2\}, \quad e_4 = \{(x, -1, x+2) : 0 \le x \le 2\}.$$

Note that $S_1 \cap \{x^2 + y^2 = \frac{1}{16}\} = A(e_4)$ and $S_1 \cap \{(x-2)^2 + y^2 = \frac{1}{16}\} = A(e_2)$, so S_1 and S_2 glue together on these edges (See Figure 2).

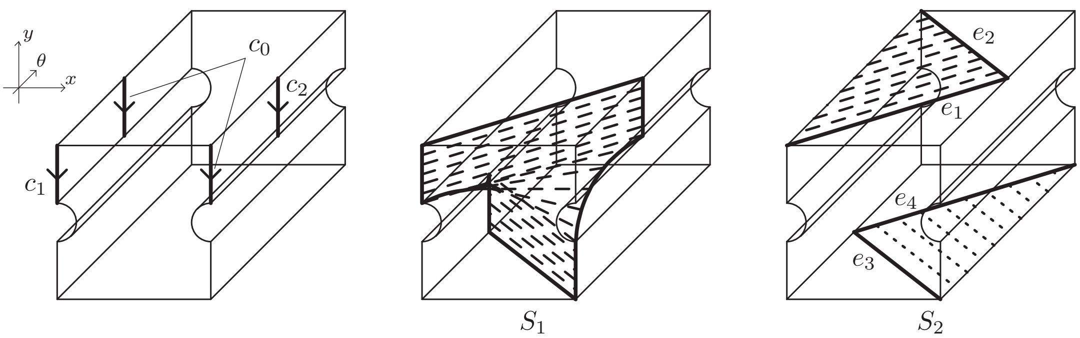

Figure 2. Birkhoff section

Since $(S_1 \cup S_2) \setminus (c_1 \cup c_2)$ is homeomorphic to the 4-punctured sphere, there exists an immersion $i_0 \colon \Sigma \to M$ of a surface Σ of genus 0 with 4 boundary components such that $i_0(\Sigma) = S_1 \cup S_2$ and $i_0(\partial\Sigma) = c_1 \cup c_2$.

Lemma 3.1. *The immersion i_0 can be deformed into a smooth one such that $i|\partial\Sigma = i_0|\partial\Sigma$ and that $i(\mathrm{Int}\Sigma)$ is an embedding transverse to the vector field Z.*

Proof. Since $\mathrm{Int}S_2$ is transverse to Z, we modify S_1 mainly. Let $S_1' = \{\theta = \sigma(x,y)\}$ be a smooth approximation of S_1 which coincides with S_1 on ∂S_1 and $\{|y| \geq \frac{1}{4}\}$ and put $K = \max_{x,y,|v|=1} |D_v\sigma(x,y)|$. We may assume that in the ϵ-neighborhood $N_\epsilon(T_2)$ of the torus $T_2 = \{x^2+y^2 = \frac{1}{16}, x \geq 0\} \cup \{(x-2)^2+y^2 = \frac{1}{16}, x \leq 2\}$ each vector Z_p is tangent to a plane $y = kx$ or $y = k(x-2)$ for some k. By Remark 2.4, we modify the vector field $Z = X+Y$ so that $|Y| > 3K|X|$ on $M_1 \setminus N_{\frac{\epsilon}{2}}(R)$, where $R = \partial M_1 \cap \{|y| < 1\}$. Then Z is transverse to $\mathrm{Int}S_1'$ on $M_1 \setminus N_{\frac{\epsilon}{2}}(R)$.

Let $\mu \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function such that:

(1) $\mu(0) = 0$ and $\mu(\xi) = 3\epsilon K$ if $\xi \geq 2\epsilon$,
(2) The graph of μ is smoothly tangent to $\{\xi = 0\}$,
(3) $\mu'(\xi)$ is non-negative and decreasing for $\xi > 0$ and $\mu'(\frac{\epsilon}{2}) = 2K$, $\mu'(\epsilon) = K$.

Define a surface $\overline{S_1}$ by $\theta = \overline{\sigma}(x,y) = \sigma(x,y) + \mu(\rho(x,y))$, where $\rho(x,y)$ is the distance between $(x,y,0)$ and R. The vector field Z is also transverse to $\overline{S_1}$ on $M_1 \setminus N_{\frac{\epsilon}{2}}(R)$, for $\max_{x,y,|v|=1} |D_v\overline{\sigma}(x,y)| \leq K + 2K = 3K$ holds there. Thus it is enough to check the transversality on $N_\epsilon(R)$.

We first see the neighborhood of T_2. Take the cylindrical coordinate (r,ω,θ) near $T_2 \cap \{x < 1\}$ so that $r = \sqrt{x^2+y^2}$ and consider an open set $\{0 < r - \frac{1}{4} < \epsilon,\ r - \frac{1}{4} \leq x\}$. Since Z is tangent to $\{\omega = \text{const.}\}$, this vector field can be described as $Z = \lambda(r,\omega)\frac{\partial}{\partial r} + \beta(r,\omega)\frac{\partial}{\partial\theta}$, where $\lambda < 0$ and $\beta > 0$. On the other hand, $\frac{\partial\overline{\sigma}}{\partial r} = \frac{\partial\sigma}{\partial r} + \mu'(r - \frac{1}{4}) > -K + K = 0$. Therefore Z is transverse to $\overline{S_1}$. Similar arguments work for the neighborhood of $T_2 \cap \{x > 1\}$.

Next we observe the region $\{0 < x < \epsilon, -1 < y < -\frac{1}{4},\ x < \sqrt{x^2+y^2} - \frac{1}{4}\}$. By construction, there holds $\overline{\sigma}(x,y) = -x + 2 + \mu(x)$, so $\frac{\partial\overline{\sigma}}{\partial x} = -1 + \mu'(x) > 0$. The vector field can be written as $Z = \eta_1(x,y)\frac{\partial}{\partial x} + \eta_2(x,y)\frac{\partial}{\partial y} + \beta(x,y)\frac{\partial}{\partial\theta}$, where $\eta_1 \leq 0, \eta_2 > 0$ and $\beta > 0$, which is transverse to $\overline{S_1}$. The transversality in the remaining part of $N_\epsilon(R)$ can be proved similarly.

Finally, let us combine the pieces $\overline{S_1}$ and S_2 to make a smooth surface. Let $(x,1,\theta)$ be a point in the neighborhood of the edge e_1. Since $\overline{S_1}$ is regarded as a graph of a function $\bar{t}(x,\theta)$ such that $\phi^{\bar{t}(x,\theta)}(x,1,\theta) \in \overline{S_1}$, we can connect it to the zero function, which gives S_2, using a partition of unity. By applying the same argument to the other edges e_2, e_3, e_4, we obtain a smooth surface transverse to Z. □

By abuse of notations, we also denote the corresponding pieces in the image $i(\Sigma)$ by S_1 and S_2.

To see that $i \colon \Sigma \to M$ is a Birkhoff section, it remains to prove that there exists a positive number t_0 so that for every point $z \in M$ the orbit $\phi^t(z)$ intersects $i(\Sigma)$ in time interval t_0. Note that for each point $(x,y,\theta) \in M_1$ the positive orbit $\{\phi^t(x,y,\theta) : t \geq 0\}$ meets T_2 if $x \neq 1$, and is attracted to γ otherwise. By construction of ϕ^t and $S_1 \cup S_2$, there exist $\epsilon > 0$ and $t_1 > 0$ such that $\phi^{[0,t_1]}(p) \cap S_1 \neq \emptyset$ for any $p \in U_\epsilon = \{|x-1| < \epsilon\}$. On the other hand, if we define $\alpha(p) = \min\{t : \phi^t(p) \in T_2\}$ for $p \in M_1 \setminus U_\epsilon$, α is bounded by the compactness of $M \setminus U_\epsilon$. Thus it is enough to see that every orbit through T intersects $i(\Sigma)$ within a bounded time interval. In fact, we can check this by tracing the orbit of ϕ^t carefully. On the torus $T_1 = \{|y| = 1\}$ take four domains $\Delta_1 = \{(x, 1, \theta) : 0 \le \theta \le x,\ 0 \le x \le 2\}$, $\Delta_2 = \{(x, 1, \theta) : -x + 4 \le \theta \le 4,\ 0 \le x \le 2\}$, $\Delta_3 = \{(x, -1, \theta) : 0 \le \theta \le -x + 2,\ 0 \le x \le 2\}$ and $\Delta_4 = \{(x, -1, \theta) : x + 2 \le \theta \le 4,\ 0 \le x \le 2\}$. Figure 3 shows the images of Δ_i's by projection on $S_1 \cup S_2$ along the orbits of ϕ^t for $t \ge 0$, where we draw $S_1 \cup S_2$ so that $S_1 \cap \gamma$ is taken to be the point at infinity. Since these projections are continuous on each Δ_i and $T = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup S_2$, every positive orbit passing through T intersects $i(\Sigma)$ within a bounded time interval.

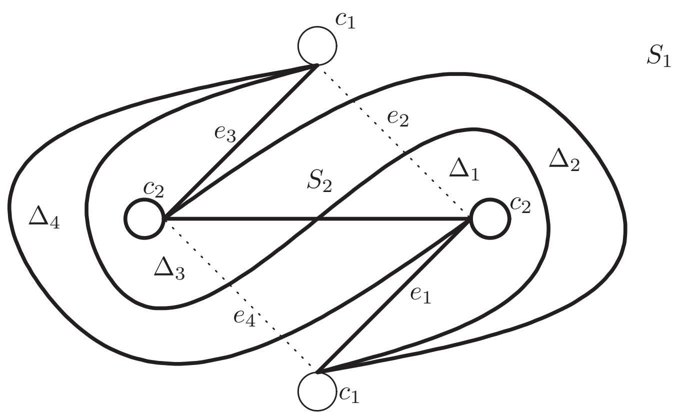

FIGURE 3. Images of Δ_i's

Next, let us describe the return map for this Birkhoff section. Let Σ^* be the surface obtained from Σ by collapsing each boundary component to a point, where Σ^* is homeomorphic to the 2-sphere, and $f^* \colon \Sigma^* \to \Sigma^*$ the pseudo-Anosov map induced from ϕ^t with invariant foliations F^u and F^s, which have 4 common singularities. Let l_k $(k = 1, 2, 3, 4)$ be the number of prongs at each singularity of F^s, then Poincaré index theorem implies that

$$2 = \chi(\Sigma^*) = \Sigma_{k=1}^4 \frac{1}{2}(2 - l_k) \le \Sigma_{k=1}^4 \frac{1}{2} = 2.$$

Thus $l_k = 1$ for $k = 1, 2, 3, 4$. (This fact can be also verified directly.) By taking a double covering with branch points being these singularities, we obtain an Anosov diffeomorphism on the 2-torus, which can be represented by a linear automorphism.

Take 4 edges e_1', e_2', e_3', e_4' on Σ^* which connect singularities of the invariant foliations and correspond to e_1, e_2, e_3, e_4 in M and denote by P_i $(i = 1, 2, 3, 4)$ the singularity within e_i' and e_{i+1}', where e_5' is understood to be e_1'. These edges divide Σ^* into two regions S_1' and S_2' corresponding to S_1 and S_2 in M. Let O be a point in S_2' such that $i(O) = i(\Sigma) \cap c_0$. Take coordinates (u, v)

on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and identify Σ^* with $T^2/(u,v) \sim (-u,-v)$ so that the points O, P_1, P_2, P_3, P_4 correspond to $(\frac{1}{4},\frac{1}{4}), (\frac{1}{2},0), (\frac{1}{2},\frac{1}{2}), (0,\frac{1}{2}), (0,0)$, respectively.

Let a_1, a_2, a_3, a_4 be oriented arcs on S_1' which start from O and end at P_1, P_2, P_3, P_4 respectively. By tracing the orbits of ϕ^t very carefully, we can see that the images of these arcs by f^* are as in Figure 4.

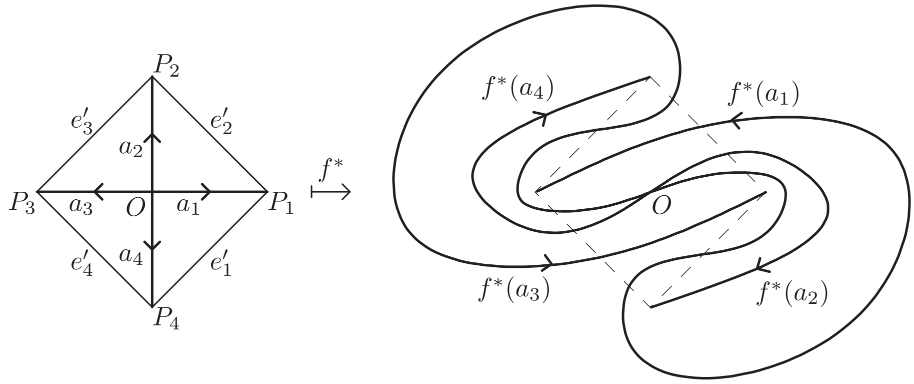

FIGURE 4. Image of arcs

Let τ_1 and τ_2 be elements in $\mathit{Diff}(\Sigma^*, \Pi)$, where $\Pi = \{P_1, P_2, P_3, P_4\}$, defined by affine maps:

$$\tau_1 \ : \ \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix},$$

$$\tau_2 \ : \ \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}.$$

See Figure 5. Since $\tau_2 \circ \tau_1 \circ f^*(a_1 \cup a_2 \cup a_3 \cup a_4)$ is ambient isotopic to $a_1 \cup a_2 \cup a_3 \cup a_4$ in $\Sigma^* \setminus \Pi$ and a diffeomorphism of $\Sigma^* \setminus (a_1 \cup a_2 \cup a_3 \cup a_4)$ relative to its boundary is isotopic to the identity, the return map f^* is isotopic to the map:

$$\tau_1^{-1} \circ \tau_2^{-1} \colon \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

Then the rigidity of pseudo-Anosov map ([FLP], Exposé 12) implies that f^* is in fact conjugate to $\tau_1^{-1} \circ \tau_2^{-1}$.

4. BRAID REPRESENTATION AND GENERALIZATION

Recall that the Bonatti-Langevin example can be reconstructed by Dehn surgeries on closed orbits of the suspension flow of f^* ([F]). Since any diffeomorphism of the 2-sphere is isotopic to the identity, the suspension flow of f^* is defined on $S^2 \times S^1$. There the closed orbits corresponding to 4 singularities make a closed braid with 4-strands and the transverse torus T is a surface bounded by that braid. Let us figure out the braid and surface in $S^2 \times S^1$.

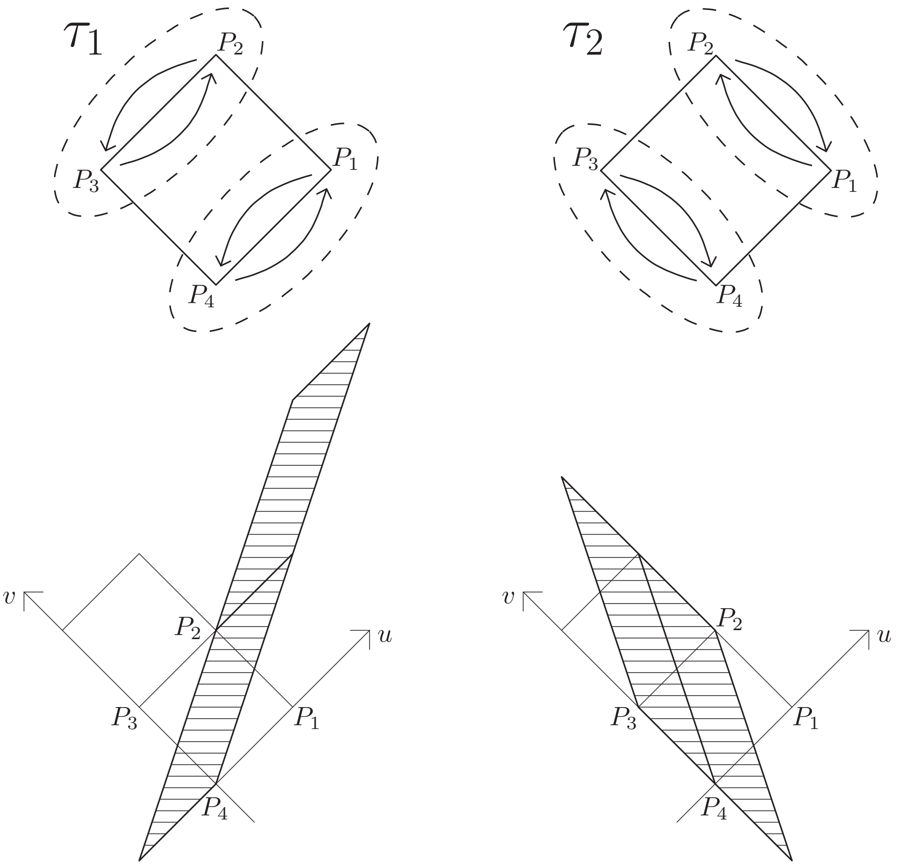

FIGURE 5. Maps τ_1 and τ_2

Take coordinates (u, v, s) on $S^2 \times S^1$ so that $(u, v) \in S^2 = T^2/(u, v) \sim (-u, -v)$ as above and $s \in S^1 = \mathbb{R}/\mathbb{Z}$. Let A_1 and A_2 be the regions in S^2 defined by $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$ and $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$ respectively. By identifying Σ^* with $S^2 \times \{0\}$, the suspension flow $\phi^t_{f^*}$ of f^* is defined on $S^2 \times S^1$. If we modify the coordinates, we may further assume that $\phi^t_{f^*}|S^2 \times \{0\}$ (resp. $\phi^t_{f^*}|S^2 \times \{\frac{1}{2}\}$) for $0 \leq t \leq \frac{1}{2}$ gives an isotopy between the identity and τ_2^{-1} (resp. τ_1^{-1}). The orbits passing through P_1, P_2, P_3, P_4 determine a closed braid which bounds the transverse torus T in $S^2 \times S^1$.

Decompose T into subsets S_2, Δ_1, Δ_2, Δ_3, Δ_4 as in Section 3 and denote $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$, then S_2 coincides with $A_1 \times \{0\}$. Note that each Δ_i share the edge e_i with S_2 and that near these edges Δ_1 and Δ_3 (resp. Δ_2 and Δ_4) lie on the past (resp. the future) of $S^2 \times \{0\}$ with respect to the time parameter t of the flow $\phi^t_{f^*}$, or equivalently the s-coordinate. Thus by an isotopy along the orbits of $\phi^t_{f^*}$, we can deform T into $h(T)$ so that $h(\Delta) \subset S^2 \times \{\frac{1}{2}\}$, $h(N_\epsilon(e_1 \cup e_3) \subset S^2 \times [\frac{1}{2}, 1]$ and $h(N_\epsilon(e_2 \cup e_4) \subset S^2 \times [0, \frac{1}{2}]$, where $N_\epsilon(\cdot)$ means an ϵ-neighborhood. On the other hand $\phi^{\frac{1}{2}}_{f^*}(h(\Delta)) = \tau_1^{-1}(h(\Delta))$ coincides the union of the images of Δ_i's in Figure 3, so $h(\Delta)$ is isotopic to the image of that union under τ_1, which is isotopic to $A_1 \times \{\frac{1}{2}\}$. Therefore we have verified that T is isotopic to the surface defined by the following (see Figure 6):

$$T' = A_1 \times \{0\} \cup A_1 \times \{\frac{1}{2}\} \cup \phi_{f*}^{[0,\frac{1}{2}]}\big((e_2 \cup e_4) \times \{0\}\big) \cup \phi_{f*}^{[0,\frac{1}{2}]}\left((e_1 \cup e_3) \times \{\frac{1}{2}\}\right).$$

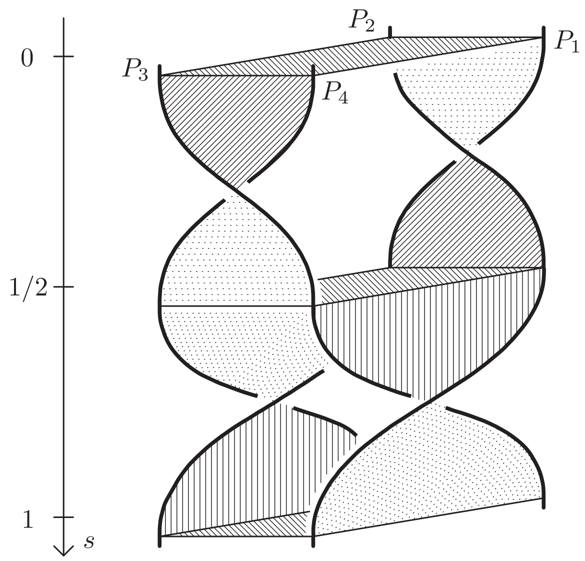

Figure 6. Closed braid and transverse torus

Indeed, this is a surface of genus 1 with 2 boundary components, each of which is tangent to a closed orbit. On a neighborhood of each boundary component, the surface T' meets the stable manifold of the corresponding closed orbit two times. Thus by the Dehn surgery along those closed orbits such that T' becomes a closed surface, we obtain an Anosov flow on the resulting manifold ([F]). This is in fact the Bonatti-Langevin example.

The description of surface by plaques and twisted bands enables us to find a family of similar examples.

Proposition 4.1. (1) *For positive odd integers n_1, n_2, the suspension flow $\phi^t_{n_1,n_2}$ of the map $\tau_1^{-n_1} \circ \tau_2^{-n_2}$ admits a genus one surface T_{n_1,n_2} bounded by two closed orbits passing through P_1, P_2, P_3, P_4.*

(2) *For positive even integers m_1, m_2, m_3, m_4, the suspension flow $\phi^t_{m_1,m_2,m_3,m_4}$ of the map $\tau_1^{-m_1} \circ \tau_2^{-m_2} \circ \tau_1^{-m_3} \circ \tau_2^{-m_4}$ admits a genus one surface T_{m_1,m_2,m_3,m_4} bounded by four closed orbits passing through P_1, P_2, P_3, P_4.*

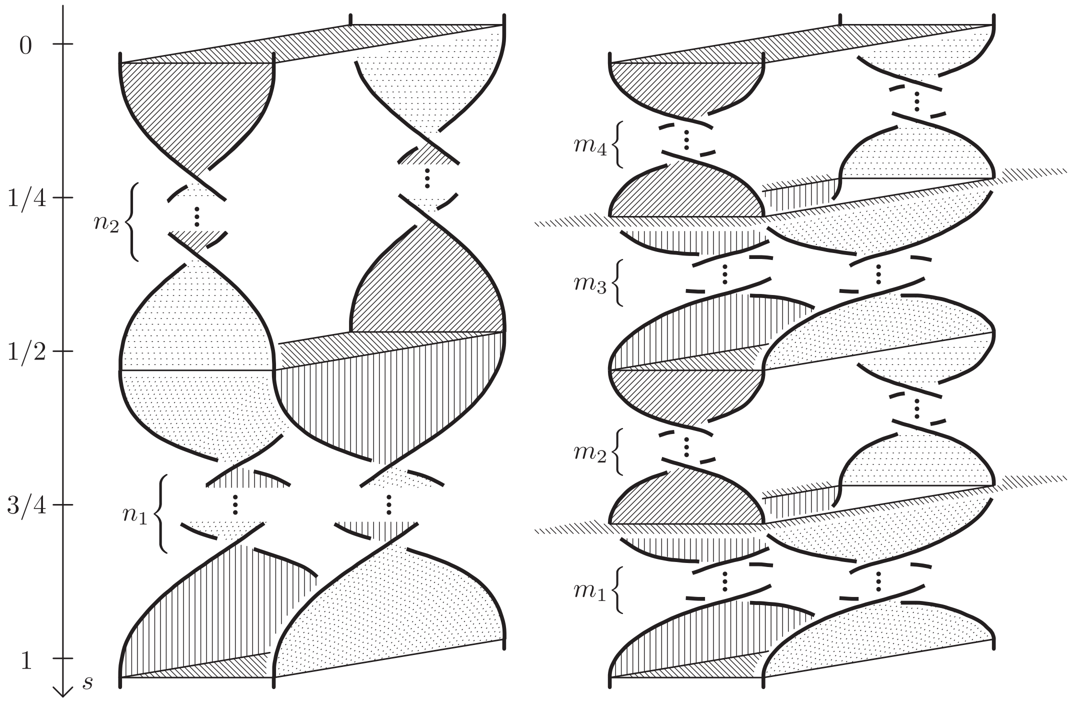

FIGURE 7. Closed braid and transverse tori for new examples

Proof. 1. We realize the flow $\phi^t_{n_1,n_2}$ as above. Then we only have to take the surface:

$$\begin{aligned} T_{n_1,n_2} &= A_1 \times \{0\} \cup A_1 \times \{\frac{1}{2}\} \cup \\ & \phi^{[0,\frac{1}{2}]}_{n_1,n_2}\big((e_2 \cup e_4) \times \{0\}\big) \cup \phi^{[0,\frac{1}{2}]}_{n_1,n_2}\left((e_1 \cup e_3) \times \{\frac{1}{2}\}\right). \end{aligned}$$

2. We realize the flow $\phi^t_{m_1,m_2,m_3,m_4}$ on $S^2 \times S^1$ so that

$$\phi^{\frac{1}{4}}(S^2 \times \{0\}) = \tau_2^{-m_4}(S^2 \times \{0\}), \quad \phi^{\frac{1}{4}}(S^2 \times \{\frac{1}{4}\}) = \tau_1^{-m_3}(S^2 \times \{\frac{1}{4}\}),$$
$$\phi^{\frac{1}{4}}(S^2 \times \{\frac{1}{2}\}) = \tau_2^{-m_2}(S^2 \times \{\frac{1}{2}\}), \quad \phi^{\frac{1}{4}}(S^2 \times \{\frac{3}{4}\}) = \tau_1^{-m_1}(S^2 \times \{\frac{3}{4}\}).$$

Define the surface:

$$\begin{aligned} T_{m_1,m_2,m_3,m_4} &= A_1 \times \{0\} \cup A_2 \times \{\frac{1}{4}\} \cup A_1 \times \{\frac{1}{2}\} \cup A_2 \times \{\frac{3}{4}\} \cup \\ & \phi^{[0,\frac{1}{4}]}_{m_1,m_2,m_3,m_4}\big((e_2 \cup e_4) \times \{0\}\big) \cup \phi^{[0,\frac{1}{4}]}_{m_1,m_2,m_3,m_4}\left((e_1 \cup e_3) \times \{\frac{1}{4}\}\right) \\ & \phi^{[0,\frac{1}{4}]}_{m_1,m_2,m_3,m_4}\left((e_2 \cup e_4) \times \{\frac{1}{2}\}\right) \cup \phi^{[0,\frac{1}{4}]}_{m_1,m_2,m_3,m_4}\left((e_1 \cup e_3) \times \{\frac{3}{4}\}\right). \end{aligned}$$

Then T_{m_1,m_2,m_3,m_4} is a surface of genus 1 with 4 boundary components. □

Theorem 4.2. *The flows $\phi^t_{n_1,n_2}$ and $\phi^t_{m_1,m_2,m_3,m_4}$ define Anosov flows by the Dehn surgeries which make the surface T_{n_1,n_2} and T_{m_1,m_2,m_3,m_4} closed surfaces respectively. Furthermore, these Anosov flows are transitive and admit a transverse torus but are not topologically conjugate to a suspension, the Bonatti-Langevin example, nor its generalization by Barbot.*

Proof. On a neighborhood of each boundary component, T_{n_1,n_2} or T_{m_1,m_2,m_3,m_4} meets twice the stable manifold of the corresponding closed orbit. Therefore by the Dehn surgeries along those closed orbits which make the torus closed, we obtain a transitive Anosov flow on the resulting manifold. Let us show that this is not conjugate to a suspension nor a BL-flow.

For a suspension Anosov flow and Anosov BL-flows, the transverse torus is determined uniquely up to isotopy. All orbits of a suspension meet the transverse torus and a BL-flow has one and only one closed orbit that avoid the transverse torus. So to see that $\phi^t_{n_1,n_2}$ or $\phi^t_{m_1,m_2,m_3,m_4}$ is not topologically conjugate to them, it is enough to prove that there exists at least two closed orbits which do not intersect T_{n_1,n_2} or T_{m_1,m_2,m_3,m_4}.

For the first case, any point $p \in S^2$ that satisfies $p \in \mathrm{Int}A_2$, $\tau_2^{-n_2}(p) \in \mathrm{Int}A_2$ and $\tau_1^{-n_1} \circ \tau_2^{-n_2}(p) = p$ defines the closed orbit passing through $p \times \{0\}$. There are $n_1 n_2$ distinct points on S^2 with this property, namely $\{(\frac{P}{4n_2}, \frac{Q}{4n_1}) : P, Q$ are odd integers such that $0 < P < 2n_2,\ 2n_1 < Q < 4n_1\}$.

To prove the latter case, let us find points $p \in S^2$ such that $p \in \mathrm{Int}A_2$, $\tau_2^{-m_4}(p) \in \mathrm{Int}A_1$, $\tau_1^{-m_3} \circ \tau_2^{-m_4}(p) \in \mathrm{Int}A_2$, $\tau_2^{-m_2} \circ \tau_1^{-m_3} \circ \tau_2^{-m_4}(p) \in \mathrm{Int}A_1$ and $\tau_1^{-m_1} \circ \tau_2^{-m_2} \circ \tau_1^{-m_3} \circ \tau_2^{-m_4}(p) = p$. This condition can be written as:

$$p \in \mathrm{Int}A_2,\ \tau_1^{m_1}(p) \in \mathrm{Int}A_1,\ \tau_2^{-m_4}(p) \in \mathrm{Int}A_1,\ \tau_2^{m_2} \circ \tau_1^{m_1}(p) = \tau_1^{-m_3} \circ \tau_2^{-m_4}(p) \in \mathrm{Int}A_2.$$

Consider the branched double-covering $\pi \colon T^2 \to S^2$ and define four squares $B_1 = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$, $B_2 = [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$, $B_3 = [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ and $B_4 = [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$. Clearly $\pi^{-1}(A_1) = B_1 \cup B_3$ and $\pi^{-1}(A_2) = B_2 \cup B_4$.

Take the point $q \in \mathrm{Int}B_4$ that is a pull-back of the point $p \in \mathrm{Int}A_2$. Since the map $\tau_1^{m_1}$ preserves the y-coordinate, $\tau_1^{m_1}(q) \in \mathrm{Int}B_1$. Similarly, $\tau_2^{m_2} \circ \tau_1^{m_1}(q) \in \mathrm{Int}B_2$, $\tau_2^{-m_4}(p) \in \mathrm{Int}B_3$ and $\tau_1^{-m_3} \circ \tau_2^{-m_4}(q) \in \mathrm{Int}B_2$. Therefore points $\tau_2^{m_2} \circ \tau_1^{m_1}(q)$ and $\tau_1^{-m_3} \circ \tau_2^{-m_4}(q)$ should be identical.

Suppose $(m_1, m_2, m_3, m_4) = (2a, 2b, 2c, 2d)$ and $q = (u_0, v_0)$. Condition $\tau_2^{m_2} \circ \tau_1^{m_1}(q) = \tau_1^{-m_3} \circ \tau_2^{-m_4}(q)$ can be written as:

$$\begin{pmatrix} 1 & 0 \\ 4b & 1 \end{pmatrix}\begin{pmatrix} 1 & 4a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \equiv \begin{pmatrix} 1 & -4c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4d & 1 \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \quad (\mathrm{mod}\ \mathbb{Z}^2)$$

or more shortly,

$$\begin{pmatrix} -16cd & 4(a+c) \\ 4(b+d) & 16ab \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (\mathrm{mod}\ \mathbb{Z}^2).$$

Now we will specify a point $q = (u_0, v_0)$ that satisfies all the conditions above. Let

$$\begin{aligned}
\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} &= \begin{pmatrix} -16cd & 4(a+c) \\ 4(b+d) & 16ab \end{pmatrix}^{-1} \begin{pmatrix} 2c+1 \\ 2b-1 \end{pmatrix} \\
&= \frac{1}{64abcd + 4(a+c)(b+d)} \begin{pmatrix} -4ab & (a+c) \\ (b+d) & 4cd \end{pmatrix} \begin{pmatrix} 2c+1 \\ 2b-1 \end{pmatrix} \\
&= \frac{1}{64abcd + 4(a+c)(b+d)} \begin{pmatrix} -8abc - 2ab + 2bc - a - c \\ 8bcd + 2bc - 2cd + b + d \end{pmatrix}.
\end{aligned}$$

Clearly $\frac{-1}{2} < u_0 < 0 < v_0 < \frac{1}{2}$, therefore $q = (u_0, v_0) \in \mathrm{Int}B_4$. Let

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 & 4a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

then $v_1 = v_0 \in (0, \frac{1}{2})$ and $u_1 = \frac{32abcd - 8acd + 2ab + 4ad + 2bc - a - c}{64abcd + 4(a+c)(b+d)} \in (0, \frac{1}{2})$, thus $\tau_1^{m_1}(q) \in \mathrm{Int}B_1$. Similarly put

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4d & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

to obtain $u_3 = u_0 \in (\frac{-1}{2}, 0)$ and $v_3 = \frac{32abcd + 8abd + 4ad + 2bc + 2cd + b + d}{64abcd + 4(a+c)(b+d)} \in (\frac{1}{2}, 1)$, which shows $\tau_2^{-m_4}(q) \in \mathrm{Int}B_3$. Finally,

$$\begin{pmatrix} u_1 \\ v_3 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 4b & 1 \end{pmatrix} \begin{pmatrix} 1 & 4a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \equiv \begin{pmatrix} 1 & -4c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4d & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \pmod{\mathbb{Z}^2},$$

therefore $\tau_2^{m_2} \circ \tau_1^{m_1}(q) = \tau_1^{-m_3} \circ \tau_2^{-m_4}(q) \in \mathrm{Int}B_2$.

Thus the closed orbit passing through $p = \pi(q) \times \{0\}$ does not meet the transverse torus. The point $q' = (u_0 + \frac{1}{2}, v_0 + \frac{1}{2})$ also satisfies similar properties and $\pi(q') \neq \pi(q)$, so it defines another closed orbit avoiding the torus. Therefore we have proved Theorem 4.2. $\square$

References

[Ba] T. Barbot, *Generalizations of the Bonatti-Langevin example of Anosov flow and their classification up to topological equivalence*, Comm. Anal. Geom, **6** (1998), 749–798

[BL] C. Bonatti and R. Langevin, *Un exemple de flot d'Anosov transitif transverse à un tore et non conjugué à une suspension*, Ergodic Theory Dynam. Systems **14** (1994), no. 4, 633–643.

[Br1] M. Brunella, *Separating the basic sets of a nontransitive Anosov flow*, Bull. London Math. Soc. **25** (1993), no. 5, 487–490.

[Br2] M. Brunella, *On the discrete Godbillon-Vey invariant and Dehn surgery on geodesic flows*, Ann. Fac. Sci. Toulouse Math. (6) **3** (1994), no. 3, 335–344.

[FLP] A. Fathi, F. Laudenbach and V. Poénaru, *Travaux de Thurston sur les surfaces*, Astérisque **66-67** (1991/1979).

[FW] J. Franks and R. Williams, *Anomalous Anosov flows*, Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979), pp. 158–174, Lecture Notes in Math., 819, Springer, Berlin, 1980.

[F] D. Fried, *Transitive Anosov flows and pseudo-Anosov maps*, Topology, **22** (1983), 299–303.

[G] S. Goodman, *Dehn surgery on Anosov flows, Geometric dynamics*, Lecture Notes in Math., 1007, 1983, Springer, Berlin, 300–307.

[HT] M. Handel and W. Thurston, *Anosov flows on new three manifolds*, Invent. Math., **59** (1980), 95–103.

[V] A. Verjovsky, *Codimension one Anosov flows*, Bol. Soc. Mat. Mexicana (2) 19 (1974), no. 2, 49–77.

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